Scalable Robustness Analysis Using Integral Quadratic Constraints

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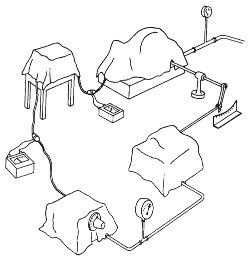
A Challenge

To Verify Performance of Complex Systems



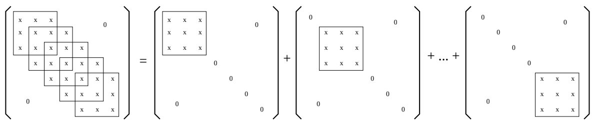
Complexity: Total model need not fit memory
Confidentiality: Component suppliers do not reveal details
Transparency: Interpretations. Design rules.

Outline

- **Positive Definite Decomposition**
- Scalable IQC analysis
- Positive and Monotone Systems
- Concluding remarks

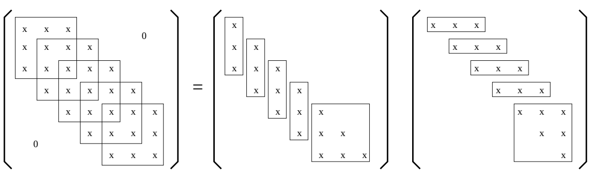
Positive Definite Decomposition

The sparse matrix on the left is positive semi-definite if and only if it can be written as a sum of positive semi-definite matrices with the structure on the right.



Proof idea

The decomposition follows immediately from the band structure of the Cholesky factors:



[Martin and Wilkinson, 1965]

Example: Vehicle formation

The first vehicle is controlled to maintain a constant speed:

$$x_1 = G_1 C_1 x_1 + w_1$$

Every other vehicle controls the distance to preceeding vehicle:

$$x_k = G_k C_k (x_{k-1} - x_k) + w_k \qquad k = 1, \dots, N$$

Is it true that $|x| \leq \gamma |w|$ for all w?

(Other requirements can be handled similarly.)

Example: Vehicle formation

$$w = \underbrace{\begin{bmatrix} 1+G_1C_1 & & & 0 \\ -G_2C_2 & \ddots & & \\ & \ddots & & \\ 0 & & -G_NC_N & 1+G_NC_N \end{bmatrix}}_{M(s)} x$$

Hence $|x| \leq \gamma |w|$ if and only if $(1+G_kC_k)^{-1}$ stable for all k and

$$M(i\omega)^* M(i\omega) - \gamma^{-2} = \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & * & * & \ddots & \\ & & \ddots & * & * \\ & & & * & * \end{bmatrix}$$

is positive semi-definite for all ω.

Example: Vehicle formation

The vehicle formation satisfies $|x| \leq \gamma |w|$ for all w if and only if there exist $K_1, \dots, K_N$ with $K_N = 0$, $K_1 = |1 + G_1C_1|^2$ such that

$$\begin{bmatrix} |G_kC_k|^2 + K_{k-1} - \gamma^{-2} & C_k^* G_k^* (1 + G_kC_k) \\ (1+G_kC_k)^* G_kC_k & |1 + G_kC_k|^2 - K_k - \gamma^{-2} \end{bmatrix} \succeq 0$$

for $k = 2, \dots, N$.

- **Complexity:** Separate test for each vehicle $2, \dots, N$.
- **Confidentiality:** Distributed seach for $K_2, \dots, K_{N-1}$.
- **Transparency:** Use H_∞ optimization to improve C_k.
- If C_k, G_k transfer functions, then K_k frequency dependent.
- Simplify K_k by model reduction.

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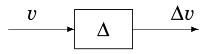
Integral Quadratic Constraint



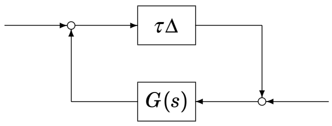
The (possibly nonlinear) operator Δ on $\mathbf{L}_2^m[0,\infty)$ is said to *satisfy the IQC defined by* Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega \geq 0$$

for all $v \in \mathbf{L}_2[0,\infty)$.

Δ structure	$\Pi(i\omega)$	Condition
Δ passive	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	
$\|\Delta(i\omega)\| \leq 1$	$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$	$x(i\omega) \geq 0$
$\delta \in [-1,1]$	$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$	$X = X^* \geq 0$, $Y = -Y^*$
$\delta(t) \in [-1,1]$	$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$	
$\Delta(s) = e^{-\theta s} - 1$	$\begin{bmatrix} x(i\omega)\rho(\omega)^2 & 0 \\ 0 & -x(i\omega) \end{bmatrix}$	$\rho(\omega) = 2\max_{\|\theta\|\leq\theta_0} \sin(\theta\omega/2)$

IQC Stability Theorem



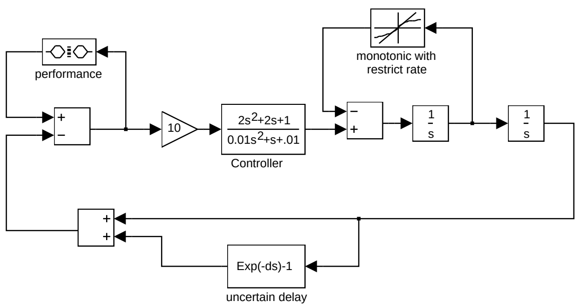
Let $G(s)$ be stable and proper and let Δ be causal.

For all $\tau \in [0,1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

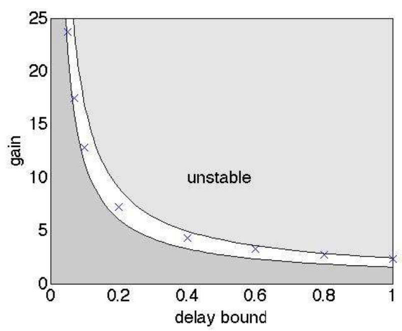
then the feedback system is input/output stable.

The IQC toolbox



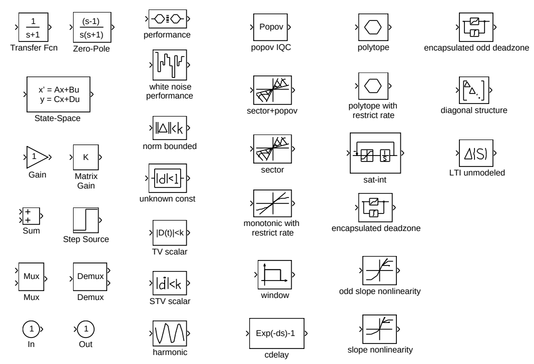

```
>> iqc_gui('fricSYSTEM')

extracting information from fricSYSTEM ...

    scalar inputs:  5
    states:         10
    simple q-forms: 7

Solving with 62 decision variables ...

  ans =     4.7139
```

Verification by IQCs



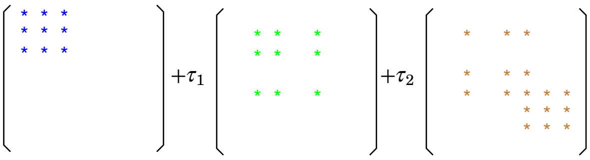
IQCs prove stability below the lower line.

A library of analysis objects



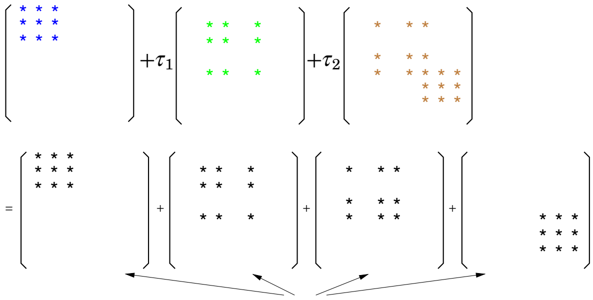
S-procedure for IQC Analysis

Find $\tau_1, \dots, \tau_n \geq 0$ such that $\sigma_0(h) + \sum_k \tau_k \sigma_k(h)$ becomes negative semi-definite:



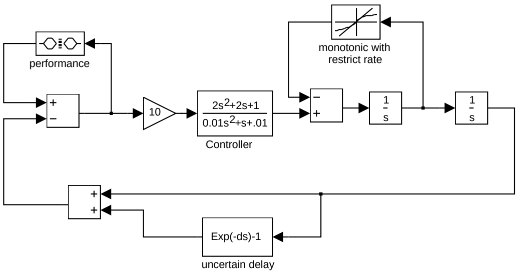
Decomposing IQC Analysis

Find $\tau_1, \dots, \tau_n \geq 0$ such that $\sigma_0(h) + \sum_k \tau_k \sigma_k(h)$ has a negative semi-definite decomposition:



Distributed certificates!

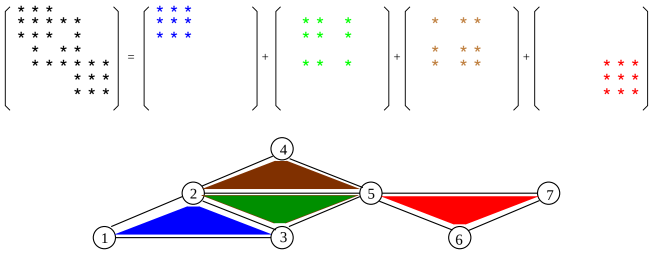
Distributed Verification



[Feron (2010)]: "The credible autocoder produces not only a target code that implements control-system specifications but also documents the target code with its properties and their proofs."

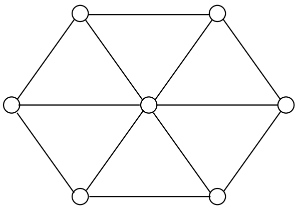
Chordal Decompositions

Cholesky factors inherit the sparsity structure of the symmetric matrix if and only if the sparsity pattern corresponds to a "chordal" graph.



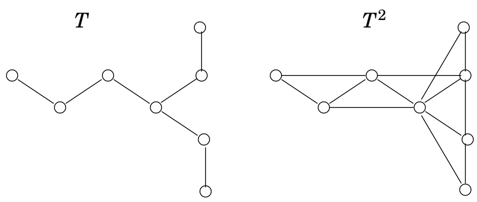
[Blair & Peyton, An introduction to chordal graphs and clique trees, 1992]

Example: Non-chordal graph



Example: Chordal graphs

If T is a tree, then T^k is chordal for every $k \geq 1$.



Theorem on Positive Extension

A matrix with entries specified according to a chordal graph has a positive definite completion if and only if all fully specified principal minors are positive definite. [Grone, et.al, 1984]

3	2	1	*	*	*	*
2	4	2	1	*	*	*
1	2	4	1	1	*	*
*	1	1	3	1	1	*
*	*	1	1	5	2	1
*	*	*	1	2	4	1
*	*	*	*	1	1	3

Network congestion control

Maximize $\sum_i U_i(x)$ over $x_i \geq 0$ subject to $\sum_i R_{li} x_i \leq c_l$

Alternatively: $\min_{p_l \geq 0} \max_{x_i \geq 0} \sum_i \left[U_i(x_i) - \sum_l p_l \left(R_{li} x_i - c_l \right) \right]$

A model for Internet dynamics can look like this:

$$\dot{x}_i(t) = k_i x_i(t) \left(1 - \frac{\sum_l R_{li} p_l(t - \tau_{li})}{U_i'(x_i(t))} \right)$$

$$\beta_l \dot{p}_l(t) + p_l(t) = \sum_i x_i(t - \tau_{il})$$

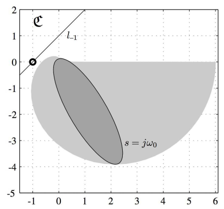
Scalable stabilty conditions by Low, Paganini, Doyle, Papachristodolou, Vinnicombe, Lestas, Pates, ...

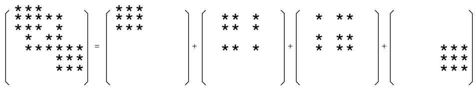
Is there a connection to scalable IQC analysis?

Network congestion control

Yes!
[Pates/Vinnicomb 2012]:

Separate the ellipse $\left\{\frac{z^*L(i\omega)z}{z^*z} : z \in \mathbb{C}^n\right\}$ from -1





Sources for conservatism: fixed decomposition
fixed separating hyperplane

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Positive systems

A linear system is called *positive* if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

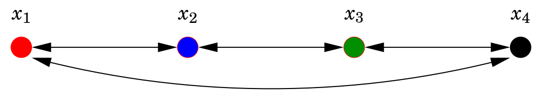
$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx$$

Equivalently, A, B and C have nonnegative coefficients except for the diagonal of A.

Examples:

- Probabilistic models.
- Economic systems.
- Chemical reactions.
- Traffic Networks.

A Scalable Stability Test



Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

. . .

Verification is scalable!

Search for Stabilizing Gains

Suppose $\begin{bmatrix} a_{11}-\ell_1 & a_{12} & 0 & a_{14} \\ a_{21}+\ell_1 & a_{22}-\ell_2 & a_{23} & 0 \\ 0 & a_{32}+\ell_2 & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \geq 0$ for $\ell_1, \ell_2 \in [0,1]$.

For stabilizing gains ℓ_1, ℓ_2, find $0 < \mu_k < \xi_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).

Examples

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics

Combination Therapy is a Control Problem

Evolutionary dynamics:

$$\dot{x} = \left(A - \sum_i u_i D^i\right) x$$

Each state x_k is the concentration of a mutant. (There can be hundreds!) Each input u_i is a drug dosage.

A describes the mutation dynamics without drugs, while $D^1, \ldots, D^m$ are diagonal matrices modeling drug effects.

Determine $u_1, \ldots, u_m \geq 0$ with $u_1 + \cdots + u_m \leq 1$ such that x decays as fast as possible!

[Jonsson, Rantzer,Murray, ACC 2014]

Optimizing Decay Rate

Stability of the matrix $A - \sum_i u_i D^i + \gamma I$ is equivalent to existence of $\xi > 0$ with

$$(A - \sum_i u_i D^i + \gamma I)\xi < 0$$

For row k, this means

$$A_k\xi - \sum_i u_i D^i_k \xi_k + \gamma\xi_k < 0$$

or equivalently

$$\frac{A_k\xi}{\xi_k} - \sum_i u_i D^i_k + \gamma < 0$$

Maximizing γ is convex optimization in $(\log \xi_i, u_i, \gamma)$!

Using Measurements of Virus Concentrations

Evolutionary dynamics:

$$\dot{x}(t) = \left(A - \sum_i u_i(t) D^i \right) x(t)$$

Can we get faster decay using time-varying $u(t)$ based on measurements of $x(t)$?

Convex Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = a$$

is a *monotone system* if its linearization is a positive system. It is a *convex monotone system* if every row of f is also convex.

Theorem. [Rantzer/ Bernhardsson (2014)]

For a convex monotone system $\dot{x} = f(x, u)$, each component of the trajectory $\phi_t(a, u)$ is a convex function of (a, u).

Using Measurements of Virus Concentrations

The evolutionary dynamics can be written as a convex monotone system:

$$\frac{d}{dt} \log x_k(t) = \frac{A_k x(t)}{x_k(t)} - \sum_i u_i(t) D_k^i$$

Hence the decay of $\log x_k$ is a convex function of the input and optimal trajectories can be found even for large systems.

Example

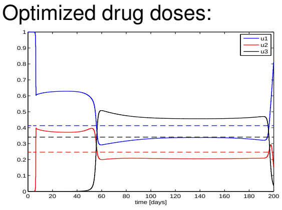
$$A = \begin{bmatrix} -\delta & \mu & \mu & 0 \\ \mu & -\delta & 0 & \mu \\ \mu & 0 & -\delta & \mu \\ 0 & \mu & \mu & -\delta \end{bmatrix}$$

clearance rate $\delta = 0.24\,\text{day}^{-1}$, mutation rate $\mu = 10^{-4}\,\text{day}^{-1}$ and replication rates for viral variants and therapies as follows

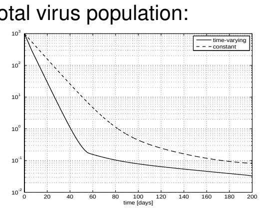
Variant	Therapy 1	Therapy 2	Therapy 3
Wild type (x_1)	$D_1^1 = 0.05$	$D_1^2 = 0.10$	$D_1^3 = 0.30$
Genotype 1 (x_2)	$D_2^1 = 0.25$	$D_2^2 = 0.05$	$D_2^3 = 0.30$
Genotype 2 (x_3)	$D_3^1 = 0.10$	$D_3^2 = 0.30$	$D_3^3 = 0.30$
HR type (x_4)	$D_4^1 = 0.30$	$D_4^2 = 0.30$	$D_4^3 = 0.15$

Example

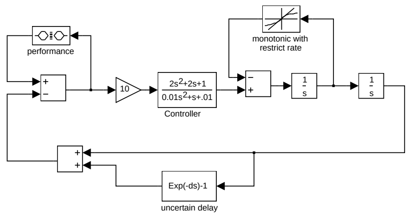
Optimized drug doses:



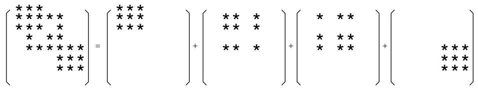
Total virus population:



Summary



IQC analysis scales using positive definite decompositions !



Scalability comes from monotonicity.