Problem 2 **(optional)**

You are given a set of identically independent data $\{x_i \in \mathbb{R}\}_{i=1}^N$ coming from an exponential distribution $\exp(\lambda)$ with parameter λ. Note that an exponential distribution $\exp(\lambda)$ has density function $p(x) = \lambda e^{-\lambda x}$.

(a) Derive the maximum likelihood (ML) estimate λ_{ML} of λ based on the observed data. Is this estimate unbiased?

A gamma distribution $\Gamma(\alpha, \beta)$ with parameters α and β has density function

$$p(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \qquad \text{with} \qquad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt\,.$$

(b) Show that if λ has a prior distribution $\lambda \sim \Gamma(\alpha, \beta)$ (before the experiment), then the posterior distribution $P(\lambda|\{x_i\}_{i=1}^N) \sim \Gamma(\alpha_{pos}, \beta_{pos})$ for some appropriate parameters α_{pos}, β_{pos}.

Hint: The posterior distribution is calculated according to Bayes Rule:

$$P(\lambda|\{x_i\}_{i=1}^N) = \frac{P(x_1, \ldots x_N|\lambda)P(\lambda)}{P(x_1, \ldots x_N)}$$

(c) Derive the maximum a posteriori (MAP) estimate λ_{MAP} of λ as a function of α and β.

Hint: The maximum a posteriori (MAP) is calculated for given data $\{x_i\}_{i=1}^N$ as $\theta_{MAP} = \text{argmax} f(x_1, \ldots x_N|\lambda) f(\lambda)$ where $f(x_1, \ldots x_N|\lambda)$ is the conditional distribution inferred from the experiment and $f(\lambda)$ is the probability density function of λ.