LUND UNIVERSITY

Optimal feedback control via cost design and control Lyapunov function

Overview

1. Optimal control
2. Angular control
3. Passivity analysis

1. Matching control
2. Linearized dynamics
3. Local stability

Motivation

Dynamic Programming (DP)

- advantages
  - \+ optimal control laws in feedback form
  - \+ well-developed theory
  - \+ necessary and sufficient for optimality
- limitations
  - \- analytical solutions for very few cases (e.g. LQR)
  - \- PDE very hard to solve
  - \- the curse of dimensionality

Optimal control: difficulty of finding a cost function

$$
\begin{aligned}
V(x_0) &= \min_u \int_0^\infty \ell(x,u)ds \\
\text{s.t.} \quad \dot{x} &= f(x,u) \\
x(0) &= x_0
\end{aligned}
$$

$$
u^*(x) = \text{argmin } V(x_0)
$$

Example

$$V(x_0) := \min_u \int_0^\infty \Big(q(x(s)) + u^\top(s) R u(s) \Big) ds,\ R = R^\top > 0$$

$$\text{s.t. } \dot{x} = -\underline{\sin}(x) + u, \quad x(0) = x_0,$$

$$x \in \mathscr{X} = \{x \in \mathbb{R}^n : ||x||_\infty < \pi/2, 1^\top \underline{\cos}(x) \geq c\},$$

for some $0 < c < n$,

$$q_1(x) = ||x||^2$$

$$q_2(x) = \underline{\sin}(x)^\top (I_n + \frac{1}{4} R^{-1})\, \underline{\sin}(x)$$

choose the cost $q_2(x)$

$$u_2^*(x) = -\frac{1}{2} R^{-1} \underline{\sin}(x)$$

$$V_2(x) = -1_n^\top \underline{\cos}(x) + n$$

Optimal control problem setup

$$\min_u \int_0^\infty \left(q(x(s)) + u(s)^\top R u(s) \right) ds$$
$$\text{s.t. } \dot{x}(t) = f(x(t)) + G^\top(x(t))\, u(t),$$
$$x(0) = x_0$$

- x the state and x_0 initial state vectors
- $R = R^\top > 0$ penalizes the input
- $q(x) > 0$ and $q(0) = 0$
- $f(x)$ is nonlinear vector field with $f(0) = 0$
- $G(x) = [g_1^\top(x), \ldots, g_m^\top(x)]^\top$ is input matrix

The idea

$\boxed{\textbf{feedback design via control Lyapunov functions}}$

feedback design via **control Lyapunov functions**

$+$

cost design from HJB/HJI

$=$

uniquely optimal feedback controllers

Lyapunov-based approach

- start from a stabilizing control law $u^*(x,R) = -\frac{1}{2} R^{-1} G(x) \nabla_x V$
- define $V : R^n \mapsto R_{>0}$ and $V(0) = 0$ (continuously differentiable) control Lyapunov function,

$$\nabla_x V^\top \Big(f(x) + G^\top(x)\, u^*(x) \Big) < -u^{*\top}(x)\, R\, u^*(x),$$

- HJB equation

$$q(x,R) + u^{*\top}(x)\, R\, u^*(x) + \nabla_x V^\top \Big(f(x) + G^\top(x)\, u^*(x) \Big) = 0$$

- design cost associated with $(u^*,\ V)$

$$q(x,R) = -\nabla_x V^\top \Big(f(x) + G^\top(x)\, u^*(x) \Big) - u^{*\top}(x)\, R\, u^*(x)$$

- R is tuning knob

$\Longrightarrow$ V is value function for $R' \leq R$ and $u^*(x,R')$ is uniquely optimal w.r.t $q(x,R')$

H_∞ — optimal control problem

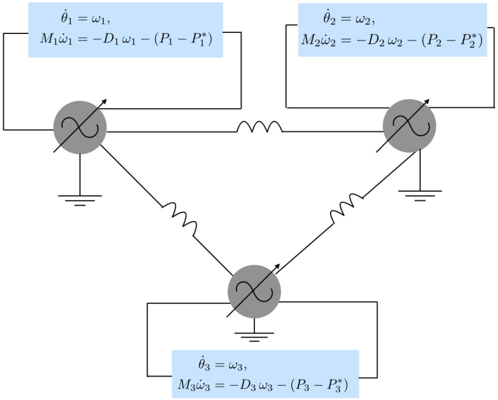
$$\begin{aligned}\min_u \max_w \int_0^\infty q(x(s)) + u(s)^\top R u(s)\, ds - \xi w^\top S w,\ \xi > 0\\ \text{s.t. } \dot{x}(t) = f(x(t)) + G^\top(x(t))u(t) + \overline{G}^\top(x(t))w(t),\\ x(0) = x_0\end{aligned}$$

- $S = S^\top > 0$ penalizes disturbance
- $\overline{G} = [\overline{g}_1^\top(x), \ldots, \overline{g}_{n_w}^\top(x)]^\top$ the disturbance input matrix
- w is system disturbance

Integration of renewables into the electrical grid

Application: angle control in converter-based power systems

Application 1: coupled oscillators



Application 1: optimization problem

$$\min_u \int_0^\infty q(\delta,\omega) + u(s)^\top R u - \xi w^\top S w \, ds$$

$$\begin{aligned} s.t. \quad \dot{\delta} &= B^\top \omega + u, \\ M\dot{\omega} &= -D\omega - B\Xi\left(\underline{\sin}(\delta) - \underline{\sin}(\delta^*)\right) + w \\ (\delta_0, \omega_0) &= (\delta(0), \omega(0)) \end{aligned}$$

- $\delta^s = B^\top \theta^s$ steady state in $\mathrm{Im}(B^\top) \cap (-\frac{\pi}{2}, \frac{\pi}{2})^3$
- $\delta = B^\top \theta$ angle differences, B incidence matrix
- ω relative frequency (to nominal), Ξ weight matrix (line susceptance)
- w represents e.g. power fluctuations

Application 1: angle control

- control Lyapunov function

$$V(\delta-\delta^s,\omega)=\frac{1}{2}\omega^\top M\omega-1_n^\top\Xi\left(\underline{\cos}(\delta)-\underline{\cos}(\delta^s)\right)-(\delta-\delta^s)^\top\Xi\,\underline{\sin}(\delta^s)$$

- optimal controller (local)

$$u^*(\delta)=-\frac{1}{2}R^{-1}\Xi\left(\underline{\sin}(\delta)-\underline{\sin}(\delta^s)\right)$$

- cost $(w=0, w\neq 0)$

$$q(\delta,\omega)=\frac{1}{4}||\underline{\sin}(\delta)-\underline{\sin}(\delta^s)||^2_{\Xi R^{-1}\Xi}+||\omega||^2_D$$
$$q(\delta,\omega)=\frac{1}{4}||\underline{\sin}(\delta)-\underline{\sin}(\delta^s)||^2_{\Xi R^{-1}\Xi}+||\omega||^2_{D-\frac{1}{4\xi}S^{-1}},\ D-\frac{1}{4\xi}S^{-1}>0$$

Simulations ($w = 0$)

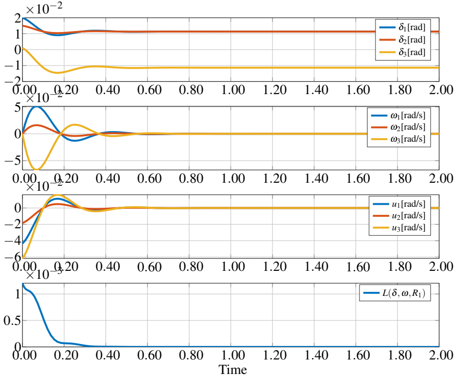


Figure: closed-loop system trajectories with R

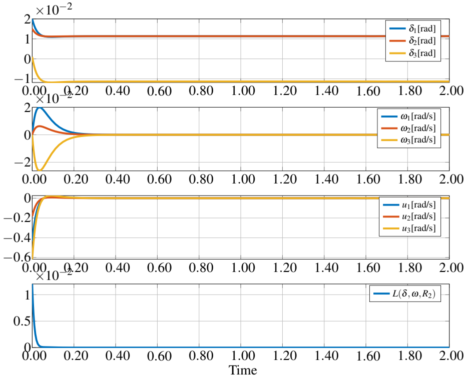


Figure: closed-loop system trajectories with $R' \leq R$

Simulations ($w \neq 0$)

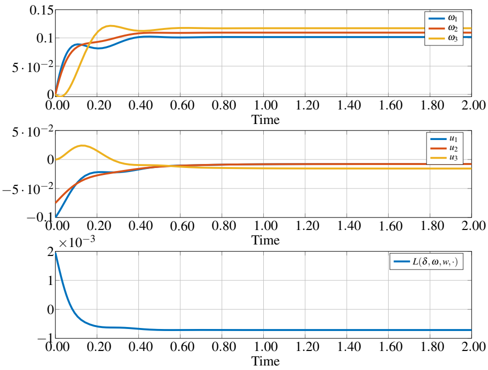


Figure: closed-loop system trajectories with non-zero disturbance w

Application 2: integrator dynamics

$$\min_u \int_0^\infty q(x(s)) + u(s)^\top R u(s) s, R > 0,$$
$$\dot{x} = u, \quad x(0) = x_0$$

Given $V(x) > 0, V(0) = 0$. The feedback controller

$$u^*(x,R) = -\frac{1}{2} R^{-1} \nabla_x V,$$

with the cost function,

$$q(x,R) = \frac{1}{4} \nabla_x V^\top R^{-1} \nabla_x V,$$

is optimal.

Application 2: angle control

$$\min_{u} \int_0^\infty \sum_{i=1}^n \left[\alpha_i u_i^2 + \frac{1}{4\alpha_i} \left(\gamma_i \tilde{\theta}_i + P_{e,i} - P_{e,i}^* \right)^2 \right],$$

$$\text{s.t. } \dot{\tilde{\theta}} = u(\tilde{\theta}),\ \tilde{\theta} \in \Omega$$

- identical DC/AC converters as controllable (virtual) angles
- set point $\theta^* = \omega^* \mathbf{1}_n t + \theta^*(0)$, $||B^\top \theta^*||_\infty < \frac{\pi}{2}$
- $\tilde{\theta} = \theta - \theta^*$ (virtual) angle difference
- power deviation $P_{e,i} - P_{e,i}^* = \sum_{j \in \mathcal{N}_i} b_{ij} \left(\sin(\tilde{\theta}_{ij} + \theta_{ij}^*) - \sin(\theta_{ij}^*) \right)$
- Ω is region of attraction

Application 2: angular droop control

let

$$R = \mathrm{diag}(\alpha_i), \quad \Gamma = \mathrm{diag}(\gamma_i)$$

$$q(x) = \sum_{i=1}^{n} \frac{1}{4\alpha_i}\left(\gamma_i \tilde{\theta}_i + P_{e,i} - P_{e,i}^*\right)^2 = \frac{1}{4}\nabla_x V^\top R^{-1} \nabla_x V$$

(control) Lyapunov function

$$V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^\top \Gamma \tilde{\theta} + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} b_{ij}\left(\cos(\tilde{\theta}_{ij} + \theta_{ij}^*) - \cos(\theta_{ij}^*) - \tilde{\theta}_{ij}\sin(\theta_{ij}^*)\right)$$

angular droop control (using PMUs)

$$u^*(\tilde{\theta}) = -\frac{1}{2\alpha_i}\left(\gamma_i \tilde{\theta}_i + \sum_{j \in \mathcal{N}_i} b_{ij}\left(\sin(\tilde{\theta}_{ij} + \theta_{ij}^*) - \sin(\theta_{ij}^*)\right)\right).$$

Application 2: closed-loop system

stability of θ^* for all angles starting in

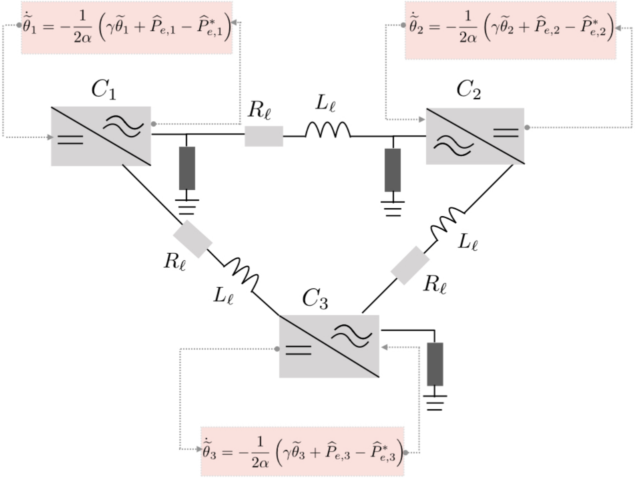
$$\Omega = \left\{ V(\tilde{\theta}) \leq l \right\} \cap \{ \tilde{\theta} \in \mathbb{R}^n, ||\tilde{\theta}||_\infty < \nu \}$$

- LQR control

$$u^*_{\mathrm{LQR}}(\tilde{\theta}) = -\frac{1}{2} R^{-1} (\Gamma + \mathscr{L}) \tilde{\theta}, \quad \mathscr{L} = B^\top \{\mathrm{diag}(b_{ij})\}_{i,j \in \mathscr{V}} B,$$

- droop properties: active power to angles $P - \tilde{\theta}$
- power sharing between converters
- no secondary control needed
- verified in prior works

Application 2: three converter system



Application 2: converter dynamics

let $U = \text{diag}(\overline{u}_1, \dots, \overline{u}_n)$.

$$\dot{\theta}_i = -\frac{1}{2\alpha_i}\left(\gamma_i(\theta_i - \theta_i^*) + \widehat{P}_{e,i} - \widehat{P}_{e,i}^*\right) + \omega^*,$$

$$\overline{u}_i = A\begin{bmatrix}\cos(\theta_i)\\ \sin(\theta_i)\end{bmatrix}, 0 < A < 1$$

DC-side dynamics,

$$C_{dc}\dot{v}_{dc} = -K_p\left(v_{dc} - v_{dc}^* 1_n\right) + U^\top i + i_{dc}^*$$

AC-side dynamics,

$$L_f \dot{i} = -R_f i + \frac{1}{2}U v_{dc} - v$$

$$C_f \dot{v} = -G_f v + i - B i_\ell$$

$$L_\ell \dot{i}_\ell = -R_\ell i_\ell + B^\top v$$

Application 2: Simulations

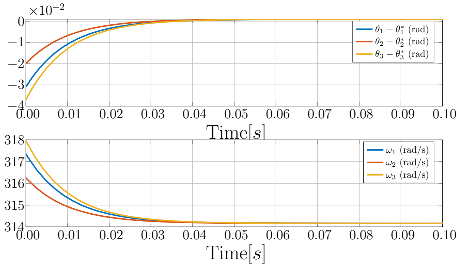


Figure: convergence of relative angle and frequencies

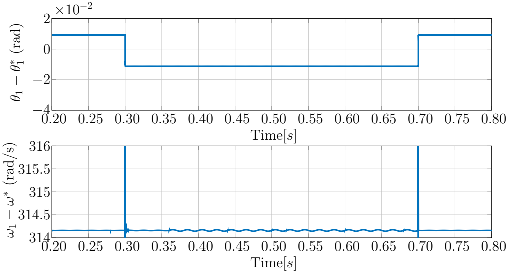


Figure: power to angle droop following a load step

Ongoing work: passivity analysis

1. study more general form of cost functions
2. apply to a class of passive systems
3. link to Port-Hamiltonian Systems

Our contributions

+ no need to solve for a value function
+ performance guarantees
+ simple control tuning similar to linear control
+ inverse optimal control
+ revisit applications of optimal control