System Modeling and Identification
Course Textbook for
FRT 041 System Identification
Dept. Automatic Control
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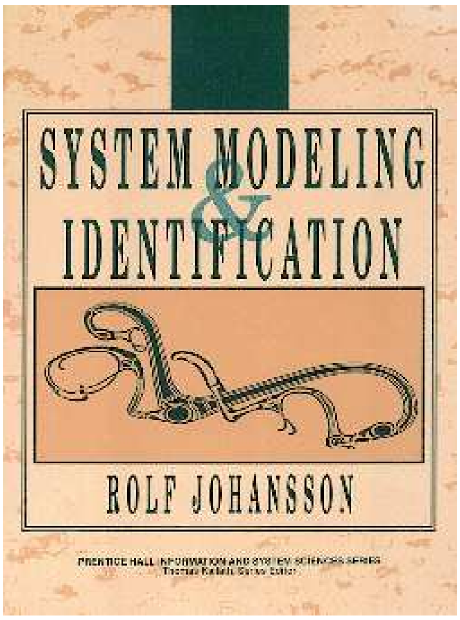

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Preface to 1st Edition

This is the solutions manual for the textbook *System Modeling and Identification*.

Several coworkers have contributed to the set of solutions presented here and I would like to thank Bo Bernhardsson, Ola Dahl, Kjell Gustafsson, Anders Hansson, Ulf Jönsson, and Henrik Olsson for their contributions in the context of the graduate course *FRT 040 System Identifiction* given at the Lund Institute of Technology.

Lund, Scandinavia, Midwinter 1992

Rolf Johansson

Preface to 2nd Edition

This is a preliminary revised version of the 1992 Solutions Manual for the textbook exercises [1]. Please report errors and suggestions for improvements to my email address Rolf.Johansson@control.lth.se.

Lund, Midwinter 2008

Rolf Johansson

Contents

2

Black-Box Models

2.1 Consider the system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) && (2.1) \\ y(t) &= Cx(t) && (2.2) \end{aligned}$$

with the input $u(t) = \delta(t)$, output y and the state $x \in \mathbb{R}^n$. Multiplication by the integrating factor e^{-At} gives

$$e^{-At}\dot{x}(t) - Ae^{-At}x(t) = e^{-At}Bu(t) \tag{2.3}$$

where the left-hand side of Eq. (2.3) can be reformulated as

$$e^{-At}\dot{x}(t) - Ae^{-At}x(t) = \frac{d}{dt}(e^{-At}x(t))$$

Integration of both hand sides of (2.3) over the time interval $[t_0, t]$ gives

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-As}Bu(s)ds$$

so that

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds$$

Evaluation of the effect of the input $u(t) = \delta(t)$ thus gives

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds && (2.4) \\ &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}B\delta(s)ds && (2.5) \\ &= e^{A(t-t_0)}x(t_0) + e^{At}B && (2.6) \end{aligned}$$

and for the output

$$y(t) = Cx(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-s)}Bu(s)ds \tag{2.7}$$

The output $y(t) = Cx(t)$ is thus

$$y(t) = Ce^{A(t-t_0)}x(t_0) + Ce^{At}B$$

which is a sum of the initial value response $Ce^{A(t-t_0)}x(t_0)$ and the impulse response $g(t) = Ce^{At}B$. The analytic expression (2.7) for the output can thus be reformulated

$$\begin{aligned} y(t) &= Cx(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-s)}Bu(s) && (2.8)\\ &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} g(t-s)u(s)ds && (2.9) \end{aligned}$$

which is on the mathematical form of a convolution.

In the case that $t_0 = 0$ and $x(t_0) = x(0) = x_0$ we thus have the initial value response $Ce^{At}x_0$.

2.2 A rectangular pulse input

$$u(t) = \begin{cases} 1/T, & 0 \le t \le T \\ 0, & t > T \text{ and } t < 0 \end{cases} \tag{2.10}$$

gives the output response

$$y(t) = \begin{cases} \dfrac{1}{T}\displaystyle\int_0^t g(\tau)d\tau, & 0 \le t \le T \\ \dfrac{1}{T}\displaystyle\int_0^t g(\tau)d\tau - \dfrac{1}{T}\displaystyle\int_0^{t-T} g(\tau)d\tau, t \ge T \end{cases} \tag{2.11}$$

If we denote the system step response by

$$s(t) = \frac{1}{T}\int_0^t g(\tau)d\tau$$

then we can express the system output as

$$y(t) = \begin{cases} s(t), & 0 \le t \le T \\ s(t) - s(t-T), & t > T \end{cases} \tag{2.12}$$

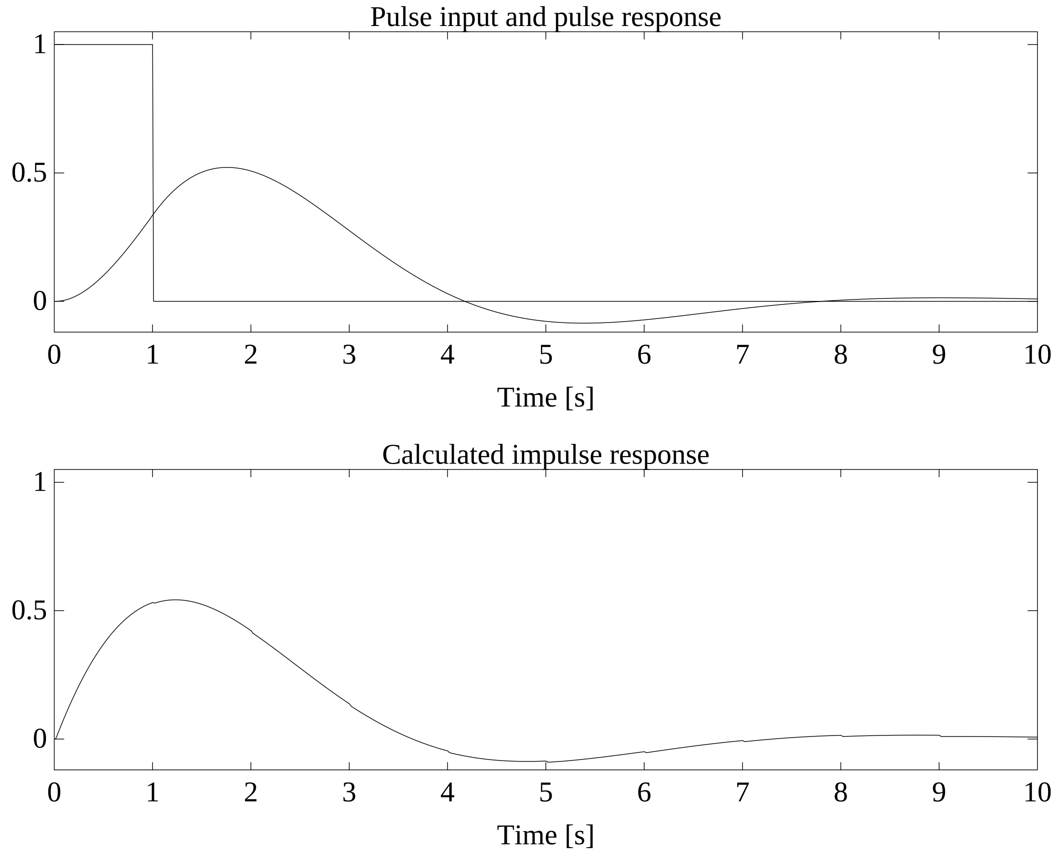

Figure 2.1 Pulse response and calculated impulse response $\widehat{g}(t)$ of a system with the transfer function $G(s) = 1/(s^2 + s + 1)$.

and we can compute the step response $s(t)$ from the output $y(t)$ as

$$s(t) = \begin{cases} y(t), & 0 \leq t \leq T \\ y(t) + s(t - T), & t > T \end{cases} \tag{2.13}$$

The impulse response $g(t)$ is then obtained as

$$g(t) = T \frac{d}{dt} s(t) \tag{2.14}$$

which has to be done by means of numerical differentiation. An approximate procedure applicable to regularly sampled data with a sampling period h can be done according to the formula

$$\widehat{g}(kh) = \frac{1}{h}(y(kh) - y((k-1)h) + \widehat{g}(kh - T) \tag{2.15}$$

2.3 We consider a case of noise corrupted frequency response analysis with an integration time $T = k \cdot (2\pi/\omega)$ for $k \in \mathbb{Z}^+$ —*i.e.*, k full periods of the test frequency sinusoid. The noise signal $v(t)$ is assumed to affect the output of the system. The goal is to determine $|G(i\omega)|$ and $\phi(\omega)$ from the system with the stationary output

$$y(t) = |G(i\omega)| \sin(\omega t + \phi(\omega)), \quad \phi(\omega) = \arg G(i\omega) \tag{2.16}$$

The output from the integrators

$$\begin{aligned} s_T &= \int_0^T (y(t) + v(t)) \sin \omega t dt = \frac{1}{2} T |G(i\omega)| \cos \phi(\omega) + \int_0^T v(t) \sin \omega t dt \\ &= \frac{1}{2} T \mathrm{Re}\ (G(i\omega)) + \Delta s_T \end{aligned} \tag{2.17}$$

$$\begin{aligned} c_T &= \int_0^T (y(t) + v(t)) \cos \omega t dt = \frac{1}{2} T |G(i\omega)| \sin \phi(\omega) + \int_0^T v(t) \cos \omega t dt \\ &= \frac{1}{2} T \mathrm{Im}\ (G(i\omega)) + \Delta c_T \end{aligned} \tag{2.18}$$

The transfer function estimate is thus

$$\widehat{G}(i\omega) = \frac{2}{T}(s_T + i c_T) = G(i\omega) + \Delta G(i\omega) \tag{2.19}$$

where the error in the transfer function estimate is

$$\Delta G(i\omega) = \frac{2}{T}(\Delta s_T + i \Delta c_T) \tag{2.20}$$

a. In the case of a sinusoidal disturbance $v(t) = A_v \sin \omega_v t$ where $\omega_v \neq \omega$ we have

$$\begin{aligned} \Delta s_T &= \int_0^T A_v \sin \omega_v t \cdot \sin \omega t dt & (2.21) \\ &= \frac{1}{2} \int_0^T A_v (\cos(\omega_v - \omega)t - \cos(\omega_v + \omega)t) dt & (2.22) \end{aligned}$$

where we have used the standard trigonometric formula $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$. Integration of (2.21) gives

$$\begin{aligned} \Delta s_T &= \frac{1}{2} A_v [\frac{\sin(\omega_v - \omega)t}{\omega_v - \omega} - \frac{\sin(\omega_v + \omega)t}{\omega_v + \omega}]_0^T = & (2.23) \\ &= \frac{1}{2} A_v (\frac{\sin(\frac{\omega_v}{\omega} 2\pi k - 2\pi k)}{\omega_v - \omega} - \frac{\sin(\frac{\omega_v}{\omega} 2\pi k + 2\pi k)}{\omega_v + \omega}) & (2.24) \\ &= \frac{A_v \omega}{\omega_v^2 - \omega^2} \sin(\frac{\omega_v}{\omega} 2\pi k), \quad \omega_v \neq \omega & (2.25) \end{aligned}$$

where we have used the circumstance that T is chosen as a full number of periods $T = k \cdot (2\pi/\omega)$ for any number $k \in \mathbb{Z}^+$.

The case of $\omega_v = \omega$ gives

$$\Delta s_T(\omega) = \int_0^T A_v \sin^2 \omega t dt = \frac{1}{2} A_v T$$

and so we can summarize for the sine channel

$$\Delta s_T(\omega) = \begin{cases} \dfrac{A_v \omega}{\omega_v^2 - \omega^2} \sin(\dfrac{\omega_v}{\omega} 2\pi k), & \omega_v \neq \omega \\ 12 A_v T, & \omega_v = \omega \end{cases} \tag{2.26}$$

Similar calculations for the cosine channel for $\omega_v \neq \omega$ gives

$$\begin{aligned} \Delta c_T &= \int_0^T A_v \sin \omega_v t \cos \omega t dt && (2.27) \\ &= \frac{1}{2} \int_0^T A_v (\sin(\omega_v - \omega)t + \sin(\omega_v + \omega)t) dt && (2.28) \\ &= \frac{1}{2} A_v [-\frac{\cos(\omega_v - \omega)t}{\omega_v - \omega} - \frac{\cos(\omega_v + \omega)t}{\omega_v + \omega}]_0^T && (2.29) \\ &= -\frac{1}{2} A_v (\frac{\cos(2\pi k \frac{\omega_v}{\omega} - 2\pi k) - 1}{\omega_v - \omega} + \frac{\cos(2\pi k \frac{\omega_v}{\omega} + 2\pi k) - 1}{\omega_v + \omega}) \\ &= A_v \frac{\omega_v}{\omega_v^2 - \omega^2} (1 - \cos(2\pi k \frac{\omega_v}{\omega})) && (2.30) \end{aligned}$$

and for $\omega_v = \omega$ we have

$$\begin{aligned} \Delta c_T &= \int_0^T A_v \sin \omega_v t \cos \omega t dt && (2.31) \\ &= \int_0^T A_v \sin \omega_v t \cos \omega_v t dt && (2.32) \\ &= \frac{1}{2} \int_0^T A_v \sin 2\omega_v t dt && (2.33) \\ &= \frac{1}{4} [A_v \cos 2\omega_v t]_0^T = 0 && (2.34) \end{aligned}$$

Thus, we can summarize for the cosine channel error

$$\Delta c_T(\omega) = \begin{cases} \dfrac{A_v \omega_v}{\omega_v^2 - \omega^2} (1 - \cos(\omega_v \omega 2\pi k)), & \omega_v \neq \omega \\ 0, & \omega_v = \omega \end{cases} \tag{2.35}$$

Hence, the transfer function error is

$$\Delta G(i\omega) = \frac{2}{T} (\Delta s_T + i \Delta c_T) \tag{2.36}$$

which decreases as $1/T$ except at $\omega_\nu = \omega$ in which case it does not help to increase the measurement duration.

Remark: Notice that a sinusoidal disturbance $\nu(t) = A_\nu \sin(\omega_\nu t + \phi)$ with a phase shift ϕ would result in somewhat different transfer function error.

b. Consider the case of white noise or high-bandwidth noise with the properties

$$\mathcal{E}\{\nu(t)\} = 0, \qquad \mathcal{E}\{\nu(t)\nu(s)\} = \sigma^2\delta(t-s) \tag{2.37}$$

Thus we have

$$\widehat{G}(i\omega) = G(i\omega) + \Delta G(i\omega), \; \widetilde{G}(i\omega) = \widehat{G}(i\omega) - G(i\omega) = \Delta G(i\omega)$$

The accuracy of the estimate is then given by the statistical properties of $\Delta G(i\omega)$.

$$\Delta G(i\omega) = \frac{2}{T}\int_0^T \nu(t)(\sin\omega t + i\cos\omega t)dt = \frac{2i}{T}\int_0^T e^{-i\omega t}\nu(t)dt \tag{2.38}$$

The mean value

$$\mathcal{E}\{\Delta\widetilde{G}(i\omega)\} = \mathcal{E}\{\Delta G(i\omega)\} = \frac{2i}{T}\int_0^T e^{-i\omega t}\mathcal{E}\{\nu(t)\}dt = 0 \tag{2.39}$$

so that the estimate of $G(i\omega)$ is unbiased

$$\mathcal{E}\{\widehat{G}(i\omega)\} = \mathcal{E}\{G(i\omega)\} = G(i\omega \tag{2.40}$$

To investigate the variance properties of a complex-valued stochastic variable x we notice that

$$\begin{aligned}
\mathrm{Var}\{x\} &= \mathcal{E}\{(x-\mathcal{E}\{x\})(x-\mathcal{E}\{x\})^*\} && (2.41)\\
&= \mathcal{E}\{xx^*\} - \mathcal{E}\{\mathcal{E}\{x\}x^*\} - \mathcal{E}\{x\mathcal{E}\{x^*\}\} + \mathcal{E}\{\mathcal{E}\{x\}\mathcal{E}\{x^*\}\}\\
&= \mathcal{E}\{xx^*\} - \mathcal{E}\{x\}\mathcal{E}\{x^*\} - \mathcal{E}\{x\}\mathcal{E}\{x^*\} + \mathcal{E}\{x\}\mathcal{E}\{x^*\}\\
&= \mathcal{E}\{xx^*\} - \mathcal{E}\{x\}\mathcal{E}\{x^*\} && (2.42)
\end{aligned}$$

Moreover, for $x = a + ib$ with mean $\mathcal{E}\{x\} = \mu_a + i\mu_b$ we have

$$\begin{aligned}
\mathrm{Var}\{x\} &= \mathcal{E}\{(a+ib)(a+ib)^*\} - (\mu_a + i\mu_b)(\mu_a + i\mu_b)^* && (2.43)\\
&= \mathcal{E}\{aa^T + bb^T) - \mu_a\mu_a^T - \mu_b\mu_b^T && (2.44)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}\{\mathrm{Re}\ x\} &= \mathcal{E}\{(a-\mu_a)(a-\mu_a)^T\} = \mathcal{E}\{aa^T\} - \mu_a\mu_a^T && (2.45)\\
\mathrm{Var}\{\mathrm{Im}\ x\} &= \mathcal{E}\{(b-\mu_b)(b-\mu_b)^T\} = \mathcal{E}\{bb^T\} - \mu_b\mu_b^T && (2.46)
\end{aligned}$$

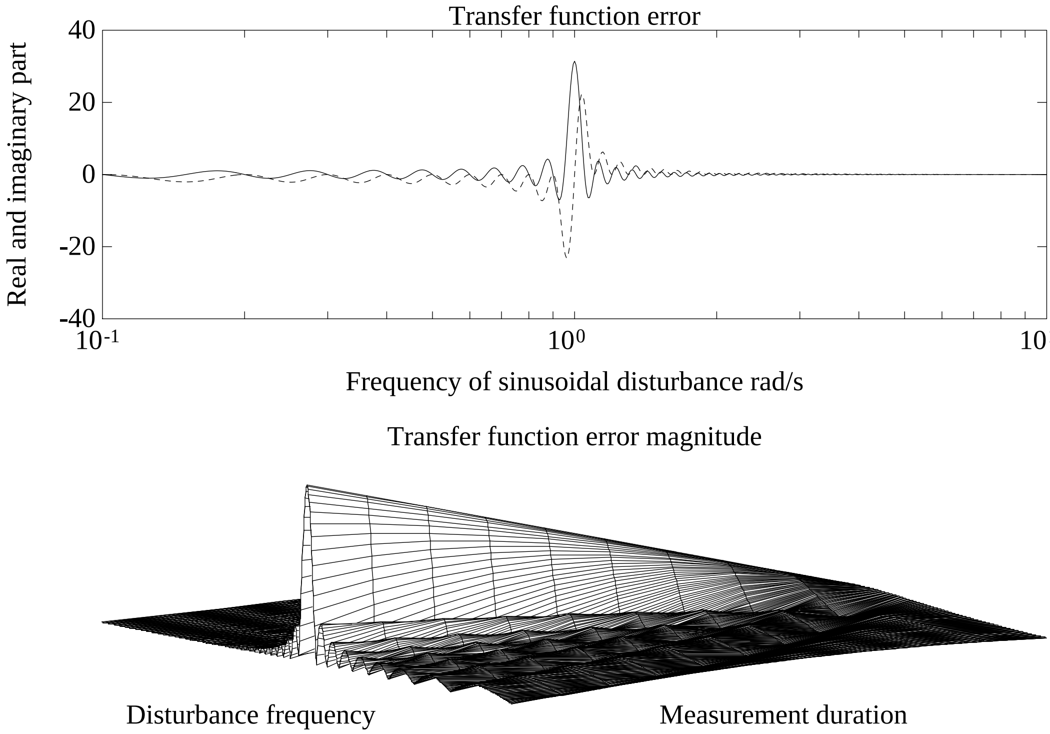

Figure 2.2 Illustration of the error of the transfer function with a test frequency $\omega = 1$ and a sinusoidal disturbance of frequency ω_v. The upper graph shows the real (*solid line*) and imaginary parts (*dashed line*) of the tranfer function error versus frequency of the disturbance. The lower graph shows the transfer function error magnitude versus frequency and versus measurement duration.

so that

$$\mathrm{Var}\{x\} \quad = \quad \mathrm{Var}\{\mathrm{Re}\ x\} + \mathrm{Var}\{\mathrm{Im}\ x\} \tag{2.47}$$

Application of this property gives

$$\begin{aligned} \mathrm{Var}\{\widehat{G}(i\omega)\} &= \mathcal{E}\{(\widehat{G}(i\omega) - \mathcal{E}\{\widehat{G}(i\omega)\})(\widehat{G}(i\omega) - \mathcal{E}\{\widehat{G}(i\omega)\})^*\} \\ &= \mathcal{E}\{\widetilde{G}(i\omega)\widetilde{G}(i\omega)^*\} = \mathrm{Var}\{\widetilde{G}(i\omega)\} \end{aligned} \tag{2.48}$$

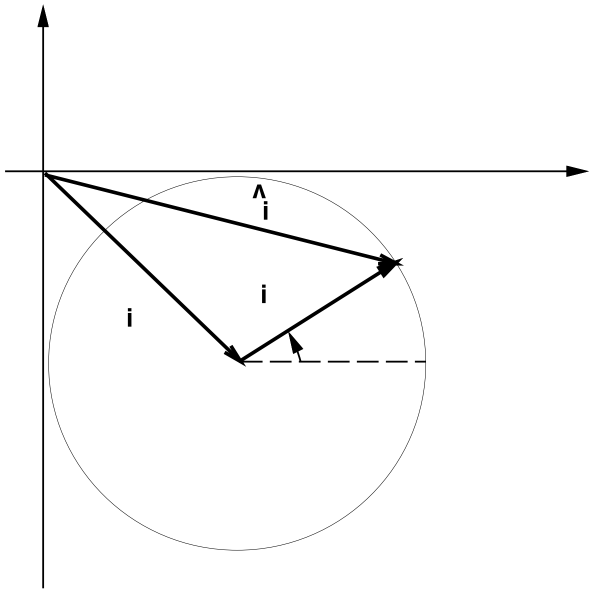

Figure 2.3 Graphical interpretation of the relationships between G, $\widetilde{G}$, $\widehat{G}$ in a polar diagram.

and further calculation gives

$$
\begin{aligned}
\mathrm{Var}\{\widehat{G}(i\omega)\} &= \mathcal{E}\{\widetilde{G}(i\omega)\widetilde{G}(i\omega)^*\} && (2.49)\\
&= \mathcal{E}\{\frac{2i}{T}\int_0^T e^{-i\omega t}v(t)dt \cdot (\frac{2i}{T}\int_0^T e^{-i\omega s}v(s)ds)^*\} && (2.50)\\
&= \frac{4}{T^2}\int_0^T\int_0^T e^{i\omega(t-s)}\mathcal{E}\{v(t)v(s)^*\}dsdt && (2.51)\\
&= \frac{4}{T^2}\int_0^T\int_0^T e^{i\omega(t-s)}\sigma^2\delta(t-s)dsdt && (2.52)\\
&= \frac{4}{T^2}\int_0^T \sigma^2 ds = \frac{4\sigma^2}{T} && (2.53)
\end{aligned}
$$

We conclude that the variance of $\widehat{G}(i\omega)$ is decreasing as $1/T$ and that $\mathcal{E}\{\widehat{G}(i\omega)\} = G(i\omega)$.

$$
\begin{aligned}
\mathcal{E}\{\widetilde{G}(i\omega)\} &= 0 && (2.54)\\
\mathrm{Var}\{\widetilde{G}(i\omega)\} &= \mathcal{E}\{\widetilde{G}(i\omega)\widetilde{G}(i\omega)^*\} = \mathcal{E}\{|\widetilde{G}(i\omega)|^2\} && (2.55)
\end{aligned}
$$

An interpretation of this result can be done according to Fig. 2.3. If we make several measurements of $G(i\omega)$, then we will have different estimates $\widehat{G}(i\omega)$ distributed around the mean $G(i\omega)$.

Consider the choice of measurement time T and frequencies when the estimate $\widehat{G}(i\omega)$ is to be used for control purposes. We have seen from

Exercise 2.3b that the accuracy of $\widehat{G}(i\omega)$ is proportional to $1/T$. In consequence, if $\mathrm{Var}\{v\}$ is large, then T needs to be large. In order to achieve a given, required accuracy of the estimate $\widehat{G}(i\omega_k)$ at a set of frequencies $\{\omega_k\}$ with the same disturbance properties, we need to apply the same measurement time T.

In addition, we need to estimate $G(i\omega)$ with higher accuracy in the frequency range where the sensitivity function has its maximum which usually appears around the cross-over frequency of the system; see Chapter 8 of *System Modeling and Identification*. Assuming that the sensitivity function is large in the frequency interval (ω_1, ω_2), then we need to make longer experiments (yielding more accurate results) in the frequency interval (ω_1, ω_2). In order to achieve a certain given measurement time $T = k \cdot (2\pi/\omega)$ we need more periods of the sinusoidal input for high frequencies than for low frequencies.

2.4 As in Exercise 2.3b we find that

$$\widehat{G}(i\omega) = G(i\omega) + \widetilde{G}(i\omega)$$

with

$$\mathcal{E}\{\widetilde{G}(i\omega)\} = 0 \tag{2.56}$$

$$\mathrm{Var}\{\widetilde{G}(i\omega)\} = \frac{4\sigma^2}{T} \tag{2.57}$$

If we consider the values $\sigma = 0.1$ and $T = 50\ s$, we can evaluate the transfer function variance as

$$\mathrm{Var}\{\widetilde{G}(i\omega)\} = 0.008$$

Considering the experimental Nyquist curve of $\widehat{G}(i\omega)$ in Fig 2.11 of the book, it is obvious that we can not exclude the possibility that the Nyquist curve of $G(i\omega)$ encircles the point -1 and that the closed-loop system $G/(1+G)$ is unstable.

As a conclusion we have that nothing can be stated about closed-loop stability and that we would need more measurements or larger T for test frequencies of the neighborhood of $\mathrm{Re}\ \widehat{G}(i\omega) = -1$—*i.e.*, where $\arg G(i\omega) \approx 180^o$.

2.5 A relay-type output nonlinearity will result in an output in the form of a square wave. According to Fourier series expansion of a square wave we have

$$y(t) = A \sum_{k=1}^{\infty} a_k \sin(k\omega t + \phi_k(\omega))$$

The contribution to the sine channel and cosine channel are

$$\begin{aligned} s_T(\omega) &= \frac{1}{2} a_1 AT \cos(\phi_1) &(2.58)\\ c_T(\omega) &= \frac{1}{2} a_1 AT \sin(\phi_1) &(2.59)\end{aligned}$$

This follows since

$$\int_0^T \sin \omega t \sin k\omega t dt = \begin{cases} T/2, & k=1 \\ 0, & k>1 \end{cases} \tag{2.60}$$

and we get the transfer function estimate

$$\widehat{G}(i\omega) = \frac{2}{T}(s_T(\omega) + ic_T(\omega)) = Aa_1 e^{i\phi(\omega)}$$

Thus we will be able to estimate the phase correctly but we will not obtain the correct gain which will appear to be constant.

2.6 According to Fig. 2.9 in the textbook we conclude that both gain and phase estimates are affected for sampled systems. Similar problems appear for discrete-time frequency response analysis. The most obvious effect of discretization is the generation of harmonics in the input. These harmonics add fortunately little to the sine and cosine channels as the integrals of $\sin \omega t \sin k\omega t$ over an interval chosen as a multiple of the test-frequency period become zero. Similar to the solution of Exercise 2.5 we expect that the frequency response $|G(i\omega)|$ might be affected. In order to avoid such problems one should recommend rapid sampling at a rate which is a multiple of the test frequency, *i.e.*, $\omega_s = n\omega$ with $n > 20$. (A detailed analysis from a signal power perspective is given in the answer to Exercise 8.1.)

2.7 Consider frequency response analysis with the system input $u(t) = u_1 \sin \omega t$. If the measurement is chosen as a multiple of half the period of the measurement frequency, *i.e.*,

$$T = k\frac{\pi}{\omega}$$

then we conclude that the integrals of the frequency response analysis are

$$\begin{aligned} c_T(\omega) &= \int_0^T y(t) \cos \omega t dt = \frac{1}{2} T|G(i\omega)|u_1 \sin \phi(\omega) &(2.61)\\ s_T(\omega) &= \int_0^T y(t) \cos \omega t dt = \frac{1}{2} T|G(i\omega)|u_1 \cos \phi(\omega) &(2.62)\end{aligned}$$

with the same expressions as in the textbook.

If the measurement time is chosen as a multiple of the full period of test frequency, *i.e.*

$$T = k\frac{2\pi}{\omega}$$

then for constant v we can find that

$$\int_0^T v \sin \omega t dt = 0$$

which means that the constant disturbance v is eliminated and does not perturb the transfer function estimate. However, using a measurement time T as a multiple of the half period will not allow for elimination a constant disturbance v from the sine channel and the cosine channel.

3

Signals and Systems

3.1 We assuming that the input $u(t)$ is constant over the sampling interval and takes on the value $u(kh)$ over the time interval $kh \leq t < (k+1)h$. If we represent the step input

$$\vartheta(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{3.1}$$

then we may represent a rectangular pulse of width h and beginning at time $t = 0$ by the rectangular pulse function

$$p(t) = \vartheta(t) - \vartheta(t-h) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t < h \\ 0, & t > h \end{cases} \tag{3.2}$$

with the z-transform

$$p(z) = (1 - z^{-1})$$

A zero-order-hold input $u(t)$ may thus be represented by

$$u(t) = \sum_{k=0}^{N} u(kh)\big(\vartheta(t-kh) - \vartheta(t-(k+1)h)\big), \quad 0 \leq t \leq Nh$$

We can represent the input as the Laplace transform and z-transform

$$\begin{aligned} U(s) &= \sum_{k=0}^{N} \frac{1}{s}(1 - e^{-sh})e^{-khs}u(kh) && (3.3) \\ U(z) &= \mathcal{Z}\{u(t)\} = \sum_{k=0}^{N} u(kh)z^{-k} && (3.4) \end{aligned}$$

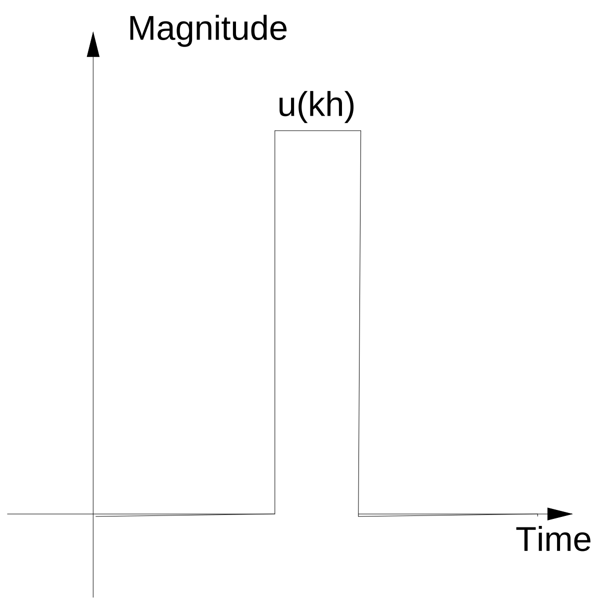

Figure 3.1 Illustration to Exercise 3.1.

and the output from a system with the transfer function $G(s)$ and the input according to Eq. (3.4) and Eq. (3.3).

$$Y(s) = G(s)U(s) = \sum_{k=1}^{N} G(s)\frac{1}{s}(1 - e^{-sh})e^{-khs}u(kh)$$

The z-transform of the output is

$$Y(z) = \mathcal{Z}\{\mathcal{L}^{-1}\{Y(s)\}\} = \mathcal{Z}\{\mathcal{L}^{-1}\{\sum_{k=0}^{N} G(s)\frac{1}{s}(1 - e^{-sh})e^{-khs}u(kh)\}\}$$

The pulse response of a transfer function $G(s)$, *i.e.*, the response to $u(t) = p(t)$ is

$$H(s) = Y(s) = G(s)U(s) = G(s)\frac{1}{s}(1 - e^{-sh})$$

or in the time domain

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{G(s)\frac{1}{s}\} * \mathcal{L}^{-1}\{(1 - e^{-sh})\}$$

The pulse transfer function is thus

$$\begin{aligned} H(z) &= \mathcal{Z}\{h(t)\} = \mathcal{Z}\{\mathcal{L}^{-1}\{(1 - e^{-sh})\}\} \cdot \mathcal{Z}\{\mathcal{L}^{-1}\{G(s)\frac{1}{s}\}\} && (3.5) \\ &= (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}\{G(s)\frac{1}{s}\}\} && (3.6) \end{aligned}$$

Example: Consider discretization of a system with the continuous-time transfer function

$$G(s) = \frac{1}{s+1}$$

Application of the procedure formulated in Exercise 3.1 gives

$$\begin{aligned}
\mathcal{Z}\{\mathcal{L}^{-1}\{G(s)\frac{1}{s}\}\} &= \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s+1}\cdot\frac{1}{s}\}\} && (3.7)\\
&= \mathcal{Z}\{\mathcal{L}^{-1}\{-\frac{1}{s+1}+\frac{1}{s}\}\} = \frac{-z}{z-e^{-h}}+\frac{z}{z-1} && (3.8)
\end{aligned}$$

and the resultant pulse transfer function is

$$H(z) = (1-z^{-1})\cdot(\frac{-z}{z-e^{-h}}+\frac{z}{z-1}) = \frac{1-e^{-h}}{z-e^{-h}} = \frac{(1-e^{-h})z^{-1}}{1-e^{-h}z^{-1}}$$

3.2 The coherence spectrum is

$$\gamma^2(\omega) = \frac{1}{1+\dfrac{S_{vv}(i\omega)}{S_{uu}(i\omega)|G(i\omega)|^2}}$$

where $S_{uu} > 0$, $S_{vv} > 0$, and $|G(i\omega)| > 0$. Hence, we immediately verify that

$$0 \le \gamma^2(\omega) \le 1, \quad \forall\omega$$

where the lower bound is obtained for a high value of the ratio of spectral densities S_{vv}/S_{uu} and the upper bound is obtained for $S_{vv}/S_{uu} = 0$.

3.3 An expression for the signal-to-noise ratio (SNR) is

$$\begin{aligned}
\text{SNR} &= \frac{e_{xx}}{e_{vv}} = \frac{e_{yy}}{e_{vv}} - 1 - \frac{e_{xv}}{e_{vv}} - \frac{e_{vx}}{e_{vv}}\\
&= \frac{e_{yy}}{e_{vv}} - 1 - 2\frac{\text{Re } e_{xv}}{e_{vv}} && (3.9)
\end{aligned}$$

in which the nonzero correlation between x and v affects the signal-to-noise ratio.

3.4 We consider a normally distributed white-noise process $\{x_k\}$ with components $x_k \in \mathcal{N}(0,\sigma^2)$ and covariance function

$$C_{xx}(\tau) = \text{Cov }\{x_k, x_{k-\tau}\} = \sigma^2\delta_{k,k-\tau}, \qquad \tau = qh, \quad q \in \mathbb{Z}$$

i.e., τ is discretized. The corresponding spectral density is

$$\begin{aligned}
S_{xx}(i\omega) &= \mathcal{F}\{C_{xx}(\tau)\} && (3.10)\\
&= h\sum_{q=-\infty}^{\infty} C_{xx}(qh)e^{-i\omega qh} && (3.11)\\
&= h\sigma^2 && (3.12)
\end{aligned}$$

which proves that the white-noise process has constant spectral density up to the Nyquist frequency $\omega_N = \pi/h$.

3.5 Consider the stochastic processes $\{x_k\}$ and $\{y_k\}$ generated by the state-space system (or Markov chain)

$$\mathcal{S}: \quad \begin{cases} x_{k+1} = \Phi x_k + \Gamma v_k \\ y_k = Cx_k + v_k \end{cases}, \quad x_k \in \mathbb{R}^n \tag{3.13}$$

where $\{v_k\}$ is a scalar white-noise process with constant spectral density

$$S_{vv}(i\omega) = h\sigma^2, \quad |\omega| \leq \omega_N = \frac{\pi}{h}$$

The output $\{y_k\}$ fulfills the input-output relationship

$$Y(z) = H(z)V(z) = \big(C(zI_{n\times n} - \Phi)^{-1}\Gamma + 1\big)V(z)$$

The output spectral density is then

$$\begin{aligned} S_{yy}(i\omega) &= H(e^{i\omega h})S_{vv}(i\omega)H^T(e^{-i\omega h}) && (3.14) \\ &= |H(e^{i\omega h})|^2 S_{vv} && (3.15) \\ &= |H(e^{i\omega h})|^2 h\sigma^2 && (3.16) \end{aligned}$$

For the state vector x_k we compute the input-output relationship

$$X(z) = H_x(z)V(z) = (zI_{n\times n} - \Phi)^{-1}\Gamma V(z)$$

Correspondingly, the spectral density of the state vector x_k is

$$\begin{aligned} S_{xx}(i\omega) &= H_x(e^{i\omega h})S_{vv}(i\omega)H_x^T(e^{-i\omega h}) && (3.17) \\ &= (e^{i\omega h}I - \Phi)^{-1}\Gamma\Gamma^T(e^{-i\omega h}I - \Phi)^{-T}\sigma^2 h && (3.18) \end{aligned}$$

3.6 The variance of the sum

$$x_1 = \widehat{\theta}_1 + \widehat{\theta}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{pmatrix} \tag{3.19}$$

is

$$\begin{aligned} \mathrm{Cov}\{x_1\} &= \mathrm{Cov}\{ \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{pmatrix} && (3.20) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \mathrm{Cov}\{\widehat{\theta}\} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (3.21) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c & -c\rho \\ -c\rho & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2c(1-\rho) && (3.22) \end{aligned}$$

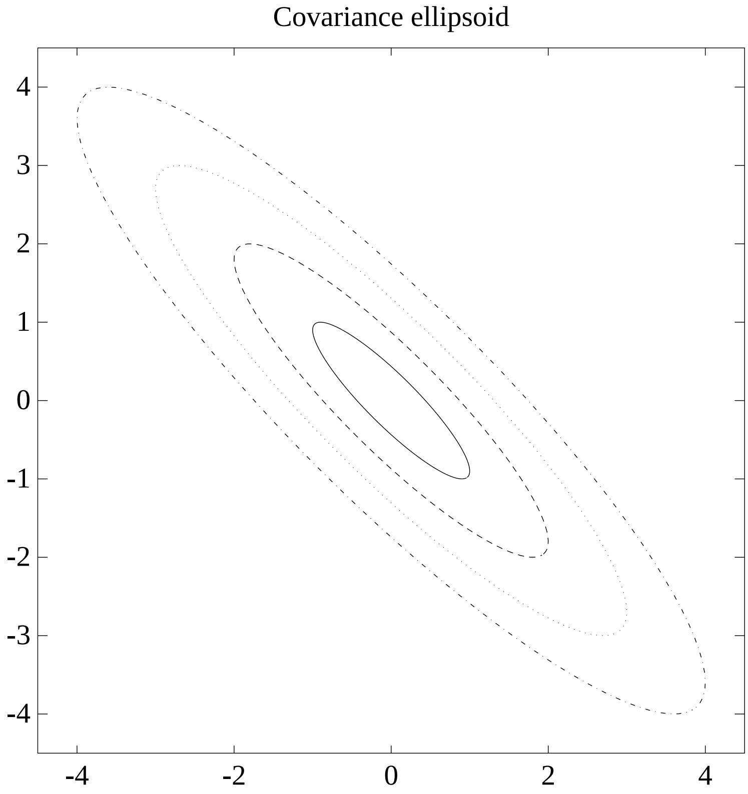

Figure 3.2 Covariance ellipsoids for the covariance matrix Σ_θ for a zero-mean stochastic variable $\widetilde{\theta}$ of Exercise 3.6 with $\rho = 0.9$.

Now introduce the variable α and consider the sum

$$x_\alpha = \widehat{\theta}_1 \cos\alpha + \widehat{\theta}_2 \sin\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \widehat{\theta},$$

with the variance

$$\begin{aligned} V(\alpha) &= \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} c & -c\rho \\ -c\rho & c \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} && (3.23) \\ &= c(\cos^2\alpha - 2\rho \sin\alpha\cos\alpha + \sin^2\alpha) && (3.24) \\ &= c(1 - \rho\sin 2\alpha) && (3.25) \end{aligned}$$

The variance $V(\alpha)$ has extrema for $\alpha = \pi/4$ and $\alpha = -\pi/4$

$$\begin{cases} \alpha = \pi/4, & V(\alpha) = c(1-\rho) \\ \alpha = -\pi/4, & V(\alpha) = c(1+\rho) \end{cases} \tag{3.26}$$

with the minimum $V(\alpha) = c(1 - |\rho|)$.

The singular value decomposition of Σ_θ is

$$\Sigma_\theta = U\Sigma V$$

with

$$U = V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \Sigma = \begin{pmatrix} c(1+\rho) & 0 \\ 0 & c(1-\rho) \end{pmatrix} \tag{3.27}$$

for $\rho > 0$ and

$$U = V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} c(1-\rho) & 0 \\ 0 & c(1+\rho) \end{pmatrix} \tag{3.28}$$

for $\rho < 0$. Thus we have that the 2-norm of Σ_θ is

$$\|\Sigma_\theta\|_2 = \sigma_1 = c(1+|\rho|)$$

which is also the maximum eigenvalue of Σ_θ. The Frobenius norm of Σ_θ is

$$\|\Sigma_\theta\|_F = \sqrt{c^2 + (c\rho)^2 + (c\rho)^2 + c^2} = \sqrt{2c^2(1+\rho^2)}$$

or

$$\|\Sigma_\theta\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{2c^2(1+\rho^2)}$$

The covariance ellipsoid is determined by the equation

$$\widetilde{\theta}^T \Sigma_\theta^{-1} \widetilde{\theta} = \text{constant}$$

or more generally the set

$$\Omega(r) = \{\widetilde{\theta} : \widetilde{\theta}^T \Sigma_\theta^{-1} \widetilde{\theta} = r^2\}, \quad r^2 = \text{constant}$$

The points belonging to the set $\Omega(r)$, *i.e*, the covariance ellipsoid, can be parametrized as follows

$$\widehat{\theta}_r = \theta + T^{-1} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \tag{3.29}$$

$$= \widetilde{\theta}_r = \widehat{\theta} - \theta = T^{-1} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \tag{3.30}$$

where T is a matrix factor of Σ_θ^{-1} in the sense that

$$T^T T = \Sigma_\theta^{-1}, \quad T = \begin{pmatrix} \dfrac{1}{\sqrt{2c(1-\rho)}} & 0 \\ 0 & \dfrac{1}{\sqrt{2c(1+\rho)}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.31}$$

For a normally distributed variable $\widetilde{\theta} \in \mathcal{N}(0, \Sigma_\theta)$ we notice that the level surfaces $\Omega(r)$ constitute the level surfaces of the probability density function

$$p(x) = \frac{1}{(2\pi \det \Sigma_\theta)^{n/2}} \exp(-\frac{1}{2} x^T \Sigma_\theta^{-1} x) = \frac{1}{(2\pi \det \Sigma_\theta)^{n/2}} \exp(-\frac{1}{2} r^2) \tag{3.32}$$

Hence the designation *covariance ellipsoid*.

3.7 Assuming that $\widetilde{\theta} \in \mathbb{R}^n$ is normally distributed $\mathcal{N}(0, \Sigma_\theta)$ with the $n \times n$ covariance matrix Σ_θ. Consider the following matrix factorization

$$\Sigma_\theta^{-1} = T^T T$$

and introduce the variables

$$\Theta = T\widetilde{\theta}, \quad \Theta \in \mathcal{N}(0, I_{n\times n}), \quad n = 2$$

A standard result, see Appendix B, is that a sum of squares of independent normally distributed variables $\{\xi_k\}_{k=1}^m$ is χ^2—distributed with m degrees of freedom, *i.e.*,

$$\chi^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2, \quad \chi^2 \in \chi^2(m)$$

Direct application of this result to the set of normally distributed variables $\Theta = T\widetilde{\theta} \in \mathcal{N}(0, I_{n\times n})$ gives the desired result

$$\Theta^T \Theta = \widetilde{\theta}^T \Sigma_\theta^{-1} \widetilde{\theta} \in \chi^2(n)$$

For a vector-valued stochastic variable, *e.g.*, a parameter error vector, of dimension n and with a covariance matrix of dimensions $n \times n$ we have

$$\mathcal{E}\{\widetilde{\theta}^T \Sigma_\theta^{-1} \widetilde{\theta}\} = \mathcal{E}\{\text{tr } \{\widetilde{\theta}^T \Sigma_\theta^{-1} \widetilde{\theta}\}\} = \mathcal{E}\{\text{tr } \{\Sigma_\theta^{-1} \widetilde{\theta}\widetilde{\theta}^T\}\} \tag{3.33}$$

$$= \text{tr } \{\Sigma_\theta^{-1} \mathcal{E}\{\widetilde{\theta}\widetilde{\theta}^T\}\} = \text{tr } \{\Sigma_\theta^{-1}\Sigma_\theta\} = \text{tr } \{I_{n\times n}\} = n \tag{3.34}$$

The covariance ellipsoid also determines the level surfaces for constant value of the probability density function

$$p(x) = \frac{1}{(2\pi \det \Sigma_\theta)^{n/2}} e^{-\frac{1}{2} x^T \Sigma_\theta^{-1} x} = \frac{1}{(2\pi \det \Sigma_\theta)^{n/2}} e^{-\frac{1}{2} r^2} \tag{3.35}$$

4

Spectrum Analysis

4.1 Let $X_1(z) = \mathcal{Z}\{x_1(k)\}$ and $X_2(z) = \mathcal{Z}\{x_2(k)\}$. The z-transform of the convolution of x_1 and x_2 is

$$\begin{aligned}
X(z) &= \mathcal{Z}\{x_1(k) * x_2(k)\} = \mathcal{Z}\{\sum_{i=-\infty}^{\infty} x_1(i)x_2(k-i)\} = && (4.1)\\
&= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x_1(i)x_2(j-i)z^{-j} = && (4.2)\\
&= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x_1(i)x_2(j-i)z^{-(j-i)-i} = && (4.3)\\
&= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x_1(i)z^{-i}x_2(j-i)z^{-(j-i)} = && (4.4)\\
&= \sum_{i=-\infty}^{\infty} x_1(i)z^{-i}\sum_{m=-\infty}^{\infty} x_2(m)z^{-m} = \mathcal{Z}\{x_1(k)\}\cdot\mathcal{Z}\{x_2(k)\} && (4.5)
\end{aligned}$$

where the last row has been obtained by changing the summation index from j to $m = j - i$.

4.2 A swept-frequency sinusoid of the type used as input can be represented by the sequence of complex exponentials

$$\{u_k\}_{k=0}^{N-1} = \{e^{i\omega_0 k^2/2}\}_{k=0}^{N-1}, \quad \text{for some constant } \omega_0$$

which is set of points located on the unit circle on the complex z-plane. The output response in the absence of an initial condition reponse is then

$$y_k = \sum_{j=0}^{N-1} h_j u_{k-j}$$

where $\{h_j\}$ represents the system transfer function $H(z)$ and

$$
\begin{aligned}
y_k &= \sum_{j=0}^{N-1} h_j u_{k-j} && (4.6)\\
&= \sum_{j=0}^{N-1} h_j e^{i\omega_0(k-j)^2/2} = && (4.7)\\
&= e^{i\omega_0 k^2/2} \sum_{j=0}^{N-1} h_j e^{i\omega_0 j^2/2} e^{-i\omega_0 kj} && (4.8)
\end{aligned}
$$

so that

$$
\frac{y_k}{u_k} = y_k e^{-i\omega_0 k^2/2} = \mathcal{Z}\{h_k e^{i\omega_0 k^2/2}\}|_{z=e^{i\omega_0 k}} \qquad (4.9)
$$

Application of the inverse discrete Fourier tranform to (4.9) gives

$$
\begin{aligned}
h'_k &= \sum_{j=1}^{N-1} \frac{y_j}{u_j} e^{i\omega_0 jk} && (4.10)\\
h_k &= h'_k e^{-i\omega_0 k^2/2} && (4.11)
\end{aligned}
$$

which determines the weighting function sequence $\{h_k\}$ of the transfer function $H(z)$ of the input-output relationship.

5

Linear Regression

5.1 Assuming that the linear regression model based on N data is

$$\mathcal{Y}_N = \Phi_N \theta + e, \quad \begin{cases} \mathcal{Y}_N \in \mathbb{R}^N \\ \Phi_N \in \mathbb{R}^{N \times p} \\ \theta \in \mathbb{R}^p \\ e \in \mathbb{R}^N \end{cases} \tag{5.1}$$

The least-squares estimate is

$$\begin{aligned} \widehat{\theta} &= (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{Y}_N && (5.2) \\ &= (\Phi_N^T \Phi_N)^{-1} \Phi_N^T (\Phi_N \theta + e) && (5.3) \\ &= \theta + (\Phi_N^T \Phi_N)^{-1} \Phi_N^T e && (5.4) \end{aligned}$$

from which we conclude that the parameter error is

$$\widetilde{\theta} = \widehat{\theta} - \theta = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T e$$

with the expectation

$$\mathcal{E}\{\widetilde{\theta}\} = \mathcal{E}\{(\Phi_N^T \Phi_N)^{-1} \Phi_N^T e\} = \mathcal{E}\{(\frac{1}{N} \Phi_N^T \Phi_N)^{-1} (\frac{1}{N} \Phi_N^T e)\}$$

which is the bias of the estimate

5.2 Consider the data

$$\mathcal{U}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \mathcal{Y}_2 = \begin{pmatrix} 6 \\ 17 \end{pmatrix} \tag{5.5}$$

generated from the system

$$\mathcal{S}: \quad y = \begin{pmatrix} 1 & u & u^2 \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & u & u^2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{5.6}$$

Assuming that the model is

$$\mathcal{M}: \qquad y = \begin{pmatrix} 1 & u & u^2 \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} \tag{5.7}$$

it is possible to organize the regressor matrix ss

$$\Phi_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \tag{5.8}$$

so that

$$\Phi_2^T \Phi_2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{pmatrix} \tag{5.9}$$

which is rank deficient. The data set in Eq. (5.5) is obviously too small for the purpose to determine all three parameters θ_0, θ_1, and θ_2.

All solutions compatible with data and the regression model $\mathcal{Y}_2 = \Phi_2\theta$ can be expressed as

$$\widehat{\theta} = \psi_1\lambda + \psi_0 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \lambda + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{5.10}$$

where λ is an arbitrary real number and where ψ_1, ψ_0 satisfy

$$\begin{aligned} \Phi_2\psi_1 &= 0 && (5.11) \\ \Phi_2\psi_0 &= \mathcal{Y}_2 && (5.12) \end{aligned}$$

If we proceed to choose the $\widehat{\theta}$ with the lowest two-norm of all minimizers, then we find the λ for which

$$\lambda^* = \arg\min_{\lambda} \|\widehat{\theta}(\lambda)\|_2 = -\frac{\psi_0^T\psi_1}{\psi_1^T\psi_1} = \frac{1}{14}$$

The corresponding solution $\widehat{\theta}$ is

$$\begin{aligned} \widehat{\theta}(\lambda^*) &= \psi_1\lambda^* + \psi_0 = (I - \frac{\psi_1\psi_1^T}{\psi_1^T\psi_1})\psi_0 && (5.13) \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \frac{1}{14} \approx \begin{pmatrix} 1.1429 \\ 1.7857 \\ 3.0714 \end{pmatrix} && (5.14) \end{aligned}$$

The same result can be obtained by means of the pseudoinverse of the matrix in Eq. (5.9)

$$(\Phi_2^T\Phi_2)^\dagger = \begin{pmatrix} 1.0816 & 0.5204 & -0.6020 \\ 0.5204 & 0.2551 & -0.2755 \\ -0.6020 & -0.2755 & 0.3776 \end{pmatrix} \tag{5.15}$$

which yields the least-squares estimate

$$\widehat{\theta} = (\Phi_2^T\Phi_2)^\dagger\Phi_2^T\mathcal{Y}_2 = \begin{pmatrix} 1.1429 \\ 1.7857 \\ 3.0714 \end{pmatrix}, \quad \|\widehat{\theta}\|_2 = 3.7321 \tag{5.16}$$

Notice that

$$\|\widehat{\theta}\|_2 < \|\theta\|_2 = \sqrt{1^2+2^2+3^2} = 3.7417$$

5.3 We consider the least-squares estimate

$$\begin{aligned} \widehat{\theta} &= \theta + (\Phi_N^T\Phi_N)^{-1}\Phi_N^T e & (5.17)\\ &= \theta + (\frac{1}{N}\Phi_N^T\Phi_N)^{-1}(\frac{1}{N}\Phi_N^T e) & (5.18) \end{aligned}$$

based on the linear regression model $y_k = \phi_k^T\theta + e_k$ where the components e_k of e are assumed to be independent and identically normally distributed variables $\mathcal{N}(0,\sigma^2)$.

From Eq. (5.17) we notice that $\widehat{\theta}$ is a sum of θ and a linear combination of the e_k's. It is well known that a a linear combination

$$z = Te, \qquad e \in \mathbb{R}^n, \quad \text{and} \quad T \in \mathbb{R}^{m\times n}$$

of normally distributed stochastic variables is also normally distributed with mean and covariance

$$\begin{aligned} \mathcal{E}\{z\} &= \mathcal{E}\{Te\} = T\mathcal{E}\{e\} = 0 & (5.19)\\ \mathcal{E}\{zz^T\} &= \mathcal{E}\{Tee^TT^T\} = T\mathcal{E}\{ee^T\}T^T = \sigma^2TT^T & (5.20) \end{aligned}$$

5.4 The weighted least-squares criterion aims to minimize the weighted sum of the squared errors between the model output and the observations.

$$V(\bar{\theta}) = \frac{1}{2}\varepsilon^TW\varepsilon = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\varepsilon_i\varepsilon_j = \frac{1}{2}(\mathcal{Y}_N - \Phi_N\bar{\theta})^TW(\mathcal{Y}_N - \Phi_N\bar{\theta}) \tag{5.21}$$

with the minimum

$$\min_{\bar{\theta}} V(\bar{\theta}) = V(\widehat{\theta})$$

obtained for the optimal estimate

$$\widehat{\theta} = (\Phi_N^T W \Phi_N)^{-1} \Phi_N^T W \mathcal{Y}_N$$

This can be seen by taking the gradient of the optimization criterion (5.21)

$$\frac{\partial V(\bar{\theta})}{\partial \bar{\theta}} = \Gamma(\bar{\theta}) = -\mathcal{Y}_N^T W \Phi_N + \bar{\theta}^T (\Phi_N^T W \Phi_N) \tag{5.22}$$

where the minimum $\partial V / \partial \bar{\theta} = 0$ provides the *normal equations*

$$\Gamma(\bar{\theta}) = -\mathcal{Y}_N^T W \Phi_N + \bar{\theta}^T (\Phi_N^T W \Phi_N) = 0 \tag{5.23}$$

The gradient $\Gamma(\bar{\theta})$ of the loss function $V(\widehat{\theta})$ takes on the value zero for $\bar{\theta} = \widehat{\theta} = (\Phi_N^T W \Phi_N)^{-1} \Phi_N^T W \mathcal{Y}_N$, *i.e.*, when $\bar{\theta}$ is chosen as the weighted least-squares estimate.

Notice that (5.22) and (5.23) are necessary conditions for obtaining a minimum. If the positive semidefinite matrix $\Phi_N^T W \Phi_N$ is assumed to be invertible, then we can also show sufficiency by completing the squares of (5.21).

$$\begin{aligned} V(\bar{\theta}) &= \frac{1}{2} (\mathcal{Y}_N - \Phi_N \bar{\theta})^T W (\mathcal{Y}_N - \Phi_N \bar{\theta}) = \\ &= \frac{1}{2} \mathcal{Y}_N^T (W - W \Phi_N (\Phi_N^T W \Phi_N)^{-1} \Phi_N^T W) \mathcal{Y}_N & (5.24) \\ &+ \frac{1}{2} \Gamma(\bar{\theta}) (\Phi_N^T W \Phi_N)^{-1} \Gamma^T(\bar{\theta}) & (5.25) \end{aligned}$$

which attains its unique minimum for $\Gamma(\bar{\theta}) = 0$, *i.e.*, for $\bar{\theta} = \widehat{\theta}$.

5.5 The prediction error based on N observations fitted to a linear regression model $\mathcal{Y}_N = \Phi_N \theta$ is

$$\varepsilon = \mathcal{Y}_N - \Phi_N \widehat{\theta}_N$$

and the orthogonality principle yields the modified equation

$$\Phi_N^T W \varepsilon = 0$$

which together give the set of linear equations

$$\begin{pmatrix} I_{N \times N} & \Phi_N \\ \Phi_N^T W & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \widehat{\theta} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_N \\ 0 \end{pmatrix} \tag{5.26}$$

However, it might be preferred to exploit the symmetric characteristics of the augmented system equation and we achieve

$$\begin{pmatrix} W^{-1} & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} \begin{pmatrix} W\varepsilon \\ \widehat{\theta} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_N \\ 0 \end{pmatrix} \tag{5.27}$$

If we introduce the weighted prediction error $\varepsilon' = W\varepsilon$, we have the augmented system equation

$$\begin{pmatrix} W^{-1} & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon' \\ \widehat{\theta} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_N \\ 0 \end{pmatrix} \tag{5.28}$$

5.6 We consider data generated by the system

$$\mathcal{S}: \qquad y_k = \phi_k^T \theta + e_k, \quad \mathcal{E}\{e_i e_j^T\} = R\delta_{ij}$$

and consider the linear regression model $\mathcal{Y}_N = \Phi_N \theta$ with

$$\mathcal{Y}_N = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \quad \text{and} \quad \Phi_N = \begin{pmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{pmatrix} \tag{5.29}$$

The residual sum is

$$\begin{aligned} V(\widehat{\theta}) &= \frac{1}{2}(\mathcal{Y}_N^T \mathcal{Y}_N - \widehat{\mathcal{Y}}_N^T \widehat{\mathcal{Y}}_N) && (5.30) \\ &= \frac{1}{2} e^T (I - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T) e && (5.31) \\ &= \frac{1}{2}\mathrm{tr}(ee^T) - \frac{1}{2}\mathrm{tr}(\Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi^T ee^T) && (5.32) \end{aligned}$$

The expected value of the residual sum is

$$\begin{aligned} \mathcal{E}\{V(\widehat{\theta})\} &= \frac{1}{2}\mathrm{tr}(\mathcal{E}\{ee^T\}) - \frac{1}{2}\mathrm{tr}(\Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{E}\{ee^T\}) && (5.33) \\ &= \frac{N}{2}\mathrm{tr}(R) - \mathrm{tr}(\Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R \end{pmatrix} && (5.34) \end{aligned}$$

In the case of a scalar R, we have the familiar result from Sec. 5.2 that

$$\begin{aligned} \mathcal{E}\{V(\widehat{\theta})\} &= \frac{N}{2}\mathrm{tr}(R) - \mathrm{tr}(\Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi^T \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & R \end{pmatrix} && (5.35) \\ &= \frac{1}{2}(N-p)R && (5.36) \end{aligned}$$

5.7 The Lagrangian associated with constrained least-squares identification presented in Sec. 5.3 of *System Modeling and Identification* is

$$L(T,\Lambda) = \widetilde{\theta}^T T^T R T \widetilde{\theta} + \mathrm{tr}\ \Lambda(T^T \Phi_N - I)$$

where Λ is a set of Lagrangian multipliers. Differentiation with respect to T shows that there is an extremum for

$$\Lambda = -2(\Phi_N^T R^{-1} \Phi_N^T)^{-1} \widetilde{\theta}\widetilde{\theta}^T$$

After elimination of Λ we have

$$\begin{aligned} L(T) &= \widetilde{\theta}^T T^T R T \widetilde{\theta} - 2\mathrm{tr}[(\Phi_N^T R^{-1} \Phi_N)^{-1} \widetilde{\theta}\widetilde{\theta}^T (T^T \Phi_N - I)] &(5.37)\\ &= \widetilde{\theta}^T T^T R T \widetilde{\theta} - 2\widetilde{\theta}^T (T\Phi_N - I)(\Phi_N^T R^{-1} \Phi_N)^{-1} \widetilde{\theta} &(5.38)\\ &= \widetilde{\theta}^T [(T - R^{-1}\Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1})^T R (T - R^{-1}\Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1})\\ &+ (\Phi_N^T R^{-1}\Phi_N)^{-1}]\widetilde{\theta} &(5.39) \end{aligned}$$

where the last step has been achieved by completing the squares of terms including the transformation matrix T. By choosing

$$\begin{aligned} T &= R^{-1}\Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1} &(5.40)\\ \widehat{\theta} &= (\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1}\mathcal{Y}_N &(5.41) \end{aligned}$$

one minimizes $L(T)$ and eliminate the first term of Eq. (5.40). The corresponding minimum value L^* of the Lagrangian $L(T)$ is

$$L^* = \widetilde{\theta}^T (\Phi_N^T R^{-1}\Phi_N)^{-1}\widetilde{\theta}$$

and

$$\begin{aligned} \widetilde{\theta} &= (\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1} e &(5.42)\\ \varepsilon &= (I - \Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1})e &(5.43) \end{aligned}$$

The residual sum of squares is

$$\begin{aligned} V(\widehat{\theta}) &= \frac{1}{2}\varepsilon(\widehat{\theta})^T\varepsilon(\widehat{\theta}) = \frac{1}{2}(\mathcal{Y}_N - \Phi_N\widehat{\theta})^T(\mathcal{Y}_N - \Phi_N\widehat{\theta}) &(5.44)\\ &= \frac{1}{2}(\Phi_N\theta + e - \Phi_N\widehat{\theta})^T(\Phi_N\theta + e - \Phi_N\widehat{\theta}) &(5.45)\\ &= \frac{1}{2}(e - \Phi_N\widetilde{\theta})^T(e - \Phi_N\widetilde{\theta}) &(5.46) \end{aligned}$$

with the expected value

$$\begin{aligned} \mathcal{E}\{V(\widehat{\theta})\} &= \mathcal{E}\{\frac{1}{2}(e - \Phi_N\widetilde{\theta})^T(e - \Phi_N\widetilde{\theta})\} &(5.47)\\ &= \mathrm{tr}(\mathcal{E}\{12(e - \Phi_N\widetilde{\theta})(e - \Phi_N\widetilde{\theta})^T\}) &(5.48)\\ &= \frac{1}{2}\mathrm{tr}(\mathcal{E}\{ee^T\} + \mathcal{E}\{\Phi_N\widetilde{\theta}\widetilde{\theta}^T\Phi_N^T\} - \mathcal{E}\{2\Phi_N\widetilde{\theta}e^T\}) &(5.49)\\ &= \frac{1}{2}\mathrm{tr}(R - \Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T) &(5.50) \end{aligned}$$

with

$$\begin{aligned}\mathcal{E}\{\widetilde{\theta}\widetilde{\theta}^T\} &= \mathcal{E}\{(\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1} e e^T R^{-1}\Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1}\} && (5.51)\\ &= (\Phi_N^T R^{-1}\Phi_N)^{-1} && (5.52)\end{aligned}$$

whereas the unconstrained estimation gives

$$\begin{aligned}\widetilde{\theta} &= (\Phi_N^T\Phi_N)^{-1}\Phi_N^T e && (5.53)\\ \varepsilon &= (I-\Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)e && (5.54)\end{aligned}$$

with the expected value

$$\begin{aligned}\mathcal{E}\{V(\widehat{\theta})\} &= \mathcal{E}\{\frac{1}{2}(e-\Phi_N\widetilde{\theta})^T(e-\Phi_N\widetilde{\theta})\} && (5.55)\\ &= \mathrm{tr}(\mathcal{E}\{\frac{1}{2}(e-\Phi_N\widetilde{\theta})(e-\Phi_N\widetilde{\theta})^T\}) && (5.56)\\ &= \frac{1}{2}\mathrm{tr}(\mathcal{E}\{ee^T\}+\mathcal{E}\{\Phi_N\widetilde{\theta}\widetilde{\theta}^T\Phi_N^T\}-\mathcal{E}\{2\Phi_N\widetilde{\theta}e^T\}) && (5.57)\\ &= \frac{1}{2}\mathrm{tr}(\mathcal{E}\{ee^T\}+\mathcal{E}\{\Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T e e^T\Phi_N(\Phi_N^T\Phi_N)^{-1}\}) \\ &- \frac{1}{2}\mathrm{tr}(\mathcal{E}\{2\Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T e e^T\}) && (5.58)\\ &= \frac{1}{2}\mathrm{tr}(R-\Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T R) && (5.59)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}\{\widetilde{\theta}\widetilde{\theta}^T\} &= \mathcal{E}\{(\Phi_N^T\Phi_N)^{-1}\Phi_N^T e e^T\Phi_N(\Phi_N^T\Phi_N)^{-1}\} && (5.60)\\ &= (\Phi_N^T\Phi_N)^{-1}\Phi_N^T R\Phi_N(\Phi_N^T\Phi_N)^{-1} && (5.61)\end{aligned}$$

We introduce the variables

$$\begin{aligned}\widehat{\theta}_{\text{constrained}} &= (\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1}\mathcal{Y}_N && (5.62)\\ \widehat{\theta}_{\text{unconstrained}} &= (\Phi_N^T\Phi_N)^{-1}\Phi_N^T\mathcal{Y}_N && (5.63)\end{aligned}$$

and

$$\begin{aligned}\Delta\theta &= \widehat{\theta}_{\text{constrained}}-\widehat{\theta}_{\text{unconstrained}} && (5.64)\\ &= (\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1}\mathcal{Y}_N-(\Phi_N^T\Phi_N)^{-1}\Phi_N^T\mathcal{Y}_N && (5.65)\\ &= [(\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1}-(\Phi_N^T\Phi_N)^{-1}\Phi_N^T](\Phi_N\theta+e) && (5.66)\\ &= (\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1}e-(\Phi_N^T\Phi_N)^{-1}\Phi_N^T e && (5.67)\end{aligned}$$

The residual sum expressed as the prediction error loss function is

$$\begin{aligned}V(\bar{\theta}) &= \frac{1}{2}(\mathcal{Y}_N-\Phi_N\bar{\theta})^T(\mathcal{Y}_N-\Phi_N\bar{\theta}) && (5.68)\\ &= \frac{1}{2}\mathcal{Y}_N^T(I-\Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)\mathcal{Y}_N && (5.69)\\ &+ \frac{1}{2}(\bar{\theta}-(\Phi_N^T\Phi_N)^{-1}\Phi_N^T\mathcal{Y}_N)^T(\Phi_N^T\Phi_N)(\bar{\theta}-(\Phi_N^T\Phi_N)^{-1}\Phi_N^T\mathcal{Y}_N)\end{aligned}$$

The prediction error loss function for the unconstrained least-squares estimate is

$$V(\widehat{\theta}_{\text{unconstrained}}) \quad = \quad \frac{1}{2}\mathcal{Y}_N^T(I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)\mathcal{Y}_N \tag{5.70}$$

whereas that for the constrained estimate is

$$\begin{aligned} V(\widehat{\theta}_{\text{constrained}}) &= \frac{1}{2}(\mathcal{Y}_N - \Phi_N\widehat{\theta}_{\text{constrained}})^T(\mathcal{Y}_N - \Phi_N\widehat{\theta}_{\text{constrained}}) \\ &= \frac{1}{2}\mathcal{Y}_N^T(I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)\mathcal{Y}_N & (5.71) \\ &+ \frac{1}{2}(\Delta\theta)^T(\Phi_N^T\Phi_N)\Delta\theta & (5.72) \end{aligned}$$

so that

$$\begin{aligned} \Delta V &= V(\widehat{\theta}_{\text{constrained}}) - V(\widehat{\theta}_{\text{unconstrained}}) & (5.73) \\ &= \frac{1}{2}\Delta\theta^T(\Phi_N^T\Phi_N)\Delta\theta \geq 0 & (5.74) \end{aligned}$$

Hence

$$\begin{aligned} V(\widehat{\theta}_{\text{constrained}}) &\geq V(\widehat{\theta}_{\text{unconstrained}}) & (5.75) \\ \mathcal{E}\{V(\widehat{\theta}_{\text{constrained}})\} &\geq \mathcal{E}\{V(\widehat{\theta}_{\text{unconstrained}})\} & (5.76) \end{aligned}$$

In order to evaluate the parameter mean-square error $\mathcal{E}\{\widehat{\theta}\widehat{\theta}^T\}$, it is necessary to check whether

$$(\Phi_N^T R^{-1}\Phi_N)^{-1} \leq (\Phi_N^T\Phi_N)^{-1}(\Phi_N^T R\Phi_N)(\Phi_N^T\Phi_N)^{-1} \tag{5.77}$$

This is equivalent to testing whether

$$(\Phi_N^T R^{-1}\Phi_N) \geq (\Phi_N^T\Phi_N)(\Phi_N^T R\Phi_N)^{-1}(\Phi_N^T\Phi_N)$$

or whether

$$R^{-1} \geq \Phi_N(\Phi_N^T R\Phi_N)^{-1}\Phi_N^T$$

or whether

$$R^{-1} - \Phi_N(\Phi_N^T R\Phi_N)^{-1}\Phi_N^T \geq 0 \tag{5.78}$$

Now let W designate the positive definite matrix

$$W = \Phi_N^T R\Phi_N \tag{5.79}$$

Using the matrix decomposition in Eq. (A.26) in *System Modeling and Identification* [1, p. 408] gives

$$\begin{pmatrix} W & \Phi_N^T \\ \Phi_N & R^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Phi_N W^{-1} & I \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & R^{-1} - \Phi_N W^{-1}\Phi_N^T \end{pmatrix} \begin{pmatrix} I & W^{-1}\Phi_N^T \\ 0 & I \end{pmatrix} \tag{5.80}$$

and resubstituting $W = \Phi_N^T R \Phi_N$ of Eq. (5.79) into the left-hand side of Eq. (5.80) gives

$$\begin{pmatrix} \Phi_N^T R\Phi_N & \Phi_N^T \\ \Phi_N & R^{-1} \end{pmatrix} = \begin{pmatrix} \Phi_N^T & 0 \\ R^{-1} & I \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi_N & R^{-1} \\ 0 & I \end{pmatrix} \geq 0 \tag{5.81}$$

which is a positive semidefinite matrix for $R = R^T > 0$. According to this calculation we ascertain that the inequalities (5.78) and (5.77) are valid for any $R = R^T > 0$ and as a result we conclude that

$$(\Phi_N^T R^{-1}\Phi_N)^{-1} \leq (\Phi_N^T\Phi_N)^{-1}(\Phi_N^T R\Phi_N)(\Phi_N^T\Phi_N)^{-1}$$

This shows that the constrained parameter estimate Eq. (5.52) has a smaller mean-square error than the unconstrained least-squares estimate Eq. (5.61).

We summarize for the prediction error and the parameter covariance

$$\mathcal{E}\{V(\widehat{\theta})\}_{\text{constrained}} \geq \mathcal{E}\{V(\widehat{\theta})\}_{\text{unconstrained}} \tag{5.82}$$

$$\mathcal{E}\{\widetilde{\theta}\widetilde{\theta}^T\}_{\text{constrained}} \leq \mathcal{E}\{\widetilde{\theta}\widetilde{\theta}^T\}_{\text{unconstrained}} \tag{5.83}$$

5.8 The augmented system matrix is

$$\begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} \tag{5.84}$$

with the inverse

$$\begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} = \begin{pmatrix} I - \Phi_N(\Phi_N^T\Phi)^{-1}\Phi_N^T & \Phi_N(\Phi_N^T\Phi_N)^{-1} \\ (\Phi_N^T\Phi_N)^{-1}\Phi_N^T & -(\Phi_N^T\Phi_N)^{-1} \end{pmatrix} \tag{5.85}$$

with $\Phi_N^T\Phi_N$ being invertible. It is easy to verify that

$$\begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} \begin{pmatrix} I - \Phi_N(\Phi_N^T\Phi)^{-1}\Phi_N^T & \Phi_N(\Phi_N^T\Phi_N)^{-1} \\ (\Phi_N^T\Phi_N)^{-1}\Phi_N^T & -(\Phi_N^T\Phi_N)^{-1} \end{pmatrix} = I_{2N\times 2N} \tag{5.86}$$

and

$$\begin{pmatrix} I - \Phi_N(\Phi_N^T\Phi)^{-1}\Phi_N^T & \Phi_N(\Phi_N^T\Phi_N)^{-1} \\ (\Phi_N^T\Phi_N)^{-1}\Phi_N^T & -(\Phi_N^T\Phi_N)^{-1} \end{pmatrix} \begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} = I_{2N\times 2N} \tag{5.87}$$

which proves the statement.

5.9 Introduce the function

$$\Gamma(\bar{\theta}) = -\mathcal{Y}_N^T R^{-1} \Phi_N + \bar{\theta}^T (\Phi_N^T R^{-1} \Phi_N)$$

By completing the squares we find

$$\begin{aligned} V(\bar{\theta}) &= \frac{1}{2}(\mathcal{Y}_N - \Phi_N \bar{\theta})^T R^{-1} (\mathcal{Y}_N - \Phi_N \bar{\theta}) && (5.88) \\ &= \frac{1}{2}\mathcal{Y}_N^T (R^{-1} - R^{-1}\Phi_N(\Phi_N^T R^{-1}\Phi_N)^{-1}\Phi_N^T R^{-1})\mathcal{Y}_N && (5.89) \\ &+ \frac{1}{2}\Gamma(\bar{\theta})(\Phi_N^T R^{-1}\Phi_N)^{-1}\Gamma^T(\bar{\theta}) && (5.90) \end{aligned}$$

which attains its unique minimum for $\Gamma(\bar{\theta}) = 0$, *i.e.*, for

$$\bar{\theta} = \widehat{\theta} = (\Phi_N^T R^{-1} \Phi_N)^{-1} \Phi_N^T R^{-1} \mathcal{Y}_N$$

5.10 Consider a multi-input, multi-output system

$$\mathcal{S}: \quad A(z^{-1})Y(z) = B(z^{-1})U(z), \quad \det A(z^{-1}) \neq 0$$

with p inputs $u_k \in \mathbb{R}^p$ and m outputs $y_k \in \mathbb{R}^m$ and time index k and polynomial matrices

$$\begin{aligned} A(z^{-1}) &= I_{m\times m} + A_1 z^{-1} + \cdots + A_n z^{-n}, \qquad A_1, \ldots, A_n \in \mathbb{R}^{m\times m} && (5.91) \\ B(z^{-1}) &= B_1 z^{-1} + \cdots + B_n z^{-n} B_1, \ldots, B_n \in \mathbb{R}^{m\times p} && (5.92) \end{aligned}$$

For the purpose of least-squares identification, then, it is suitable to organize model and data according to

$$\begin{aligned} y_k &= -A_1 y_{k-1} - \cdots - A_n y_{k-n} + B_1 u_{k-1} + \cdots + B_n u_{k-n} && (5.93) \\ \phi_k &= \begin{pmatrix} -y_{k-1}^T & \ldots & -y_{k-n}^T & u_{k-1}^T & \ldots & u_{k-n}^T \end{pmatrix}^T && (5.94) \end{aligned}$$

with $y_k \in \mathbb{R}^m$ and $\phi_k \in \mathbb{R}^{n(m+p)}$. The parameter matrix

$$\theta = \begin{pmatrix} A_1^T \\ \vdots \\ A_n^T \\ B_1^T \\ \vdots \\ B_n^T \end{pmatrix}, \quad \theta \in \mathbb{R}^{n(m+p)\times m} \qquad (5.95)$$

which suggests the linear regression model

$$\mathcal{M}: \quad \mathcal{Y} = \Phi\theta, \quad \text{with} \quad \mathcal{Y} = \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_N^T \end{pmatrix}, \quad \text{and} \quad \Phi = \begin{pmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{pmatrix} \tag{5.96}$$

with $\mathcal{Y} \in \mathbb{R}^{N\times m}$ and $\Phi \in \mathbb{R}^{N\times n(m+p)}$. Now introduce the following matrix decompositions into column vectors

$$\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_{:1} & \mathcal{Y}_{:2} & \dots & \mathcal{Y}_{:m} \end{pmatrix} \tag{5.97}$$

$$\Phi = \begin{pmatrix} \Phi_{:1} & \Phi_{:2} & \dots & \Phi_{:m} \end{pmatrix} \tag{5.98}$$

$$\varepsilon = \begin{pmatrix} \varepsilon_{:1} & \varepsilon_{:2} & \dots & \varepsilon_{:m} \end{pmatrix} \tag{5.99}$$

$$\theta = \begin{pmatrix} \theta_{:1} & \theta_{:2} & \dots & \theta_{:m} \end{pmatrix} \tag{5.100}$$

Let $\bar{\theta}_{:j}, \quad j = 1, 2, \dots, m$ denote an arbitrary estimate of the parameter vector $\theta_{:j}$. The least-squares criterion aims to minimize the sum of the squared errors between the model output and the observations.

$$V_j(\bar{\theta}_{:j}) = \frac{1}{2}\varepsilon_{:j}^T\varepsilon_{:j} = \frac{1}{2}(\mathcal{Y}_{:j} - \Phi\bar{\theta}_{:j})^T(\mathcal{Y}_{:j} - \Phi\bar{\theta}_{:j}) \tag{5.101}$$

each V_j with the minimum

$$\min_{\bar{\theta}_{:j}} V_j(\bar{\theta}_{:j}) = V_i(\widehat{\theta}_{:j})$$

obtained for the optimal estimate $\bar{\theta}_{:j} = \widehat{\theta}_{:j}$. By taking the gradient of the optimization criterion (5.101), we have

$$0 = \frac{\partial V_j(\bar{\theta}_{:j})}{\partial \bar{\theta}_{:j}} = -\mathcal{Y}_{:j}^T\Phi + \bar{\theta}_{:j}^T(\Phi^T\Phi) \tag{5.102}$$

where the minimum $\partial V_i/\partial\bar{\theta} = 0$ provides the normal equations

$$-\Phi^T\mathcal{Y}_{:j} + (\Phi^T\Phi)\bar{\theta}_{:j} = 0, \qquad i = 1, 2, \dots, m \tag{5.103}$$

for each parameter vector $\theta_{:j}, \quad i = 1, 2, \dots, m$. By arranging these equations column-wise, we have the set of normal equations

$$-\Phi^T \begin{pmatrix} \mathcal{Y}_{:1} & \mathcal{Y}_{:2} & \dots & \mathcal{Y}_{:m} \end{pmatrix} + (\Phi^T\Phi)\begin{pmatrix} \bar{\theta}_{:1} & \bar{\theta}_{:2} & \dots & \bar{\theta}_{:m} \end{pmatrix} = 0 \tag{5.104}$$

or

$$-\Phi^T \mathcal{Y} + (\Phi^T \Phi)\bar{\theta} = 0$$

which is a suitable formulation of the matrix normal equations. The normal equations of the associated least-squares estimation of θ will, as a result of the non-uniqueness of parameters, in general exhibit rank deficit. It is therefore natural to apply the least-squares solution

$$\widehat{\theta} = (\Phi^T \Phi)^{\dagger} \Phi^T \mathcal{Y}$$

where $(\Phi_N^T \Phi_N)^{\dagger}$ denotes the matrix pseudo-inverse of $\Phi^T \Phi$; see Appendix A of *System Modeling and Identification*. The associated least-squares estimate then obtained has the smallest 2-norm of all possible minimizers of the least-squares criterion.

From the properties of the least-squares solution, we have

$$\mathcal{Y}_N = \Phi_N \theta + e_N, \quad \widehat{\theta} = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{Y}_N \tag{5.105}$$

so that

$$\begin{aligned}
\widehat{\mathcal{Y}}_N &= \Phi_N \widehat{\theta} = \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{Y}_N && (5.106)\\
\varepsilon_N &= \mathcal{Y}_N - \widehat{\mathcal{Y}}_N = (I_N - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T) \mathcal{Y}_N && (5.107)\\
&= (I_N - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T)(\Phi_N \theta + e_N) && (5.108)\\
&= (I_N - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T) e_N && (5.109)\\
\varepsilon_N^T \varepsilon_N &= (\mathcal{Y}_N - \widehat{\mathcal{Y}}_N)^T (\mathcal{Y}_N - \widehat{\mathcal{Y}}_N) && (5.110)\\
&= \mathcal{Y}_N^T (I_N - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T) \mathcal{Y}_N && (5.111)\\
&= e_N^T (I_N - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T) e_N && (5.112)
\end{aligned}$$

The residual sum of squares is

$$\begin{aligned}
V(\widehat{\theta}) &= \sum_{j=1}^{m} V(\widehat{\theta}_{:j}) = \mathrm{tr}\{\frac{1}{2}(\mathcal{Y}^T \mathcal{Y} - \widehat{\mathcal{Y}}^T \widehat{\mathcal{Y}})\} && (5.113)\\
&= \mathrm{tr}\{\frac{1}{2} e^T (I - \Phi_N (\Phi_N^T \Phi_N)^{-1} \Phi_N^T) e && (5.114)
\end{aligned}$$

5.11 It is straightforward to adopt the first-order linear regression model

$$\mathcal{M}: \quad \begin{cases} y_k = \phi_k^T \theta \\ \mathcal{Y}_N = \Phi_N \theta \end{cases} \quad \text{with } \theta = \begin{pmatrix} a \\ b \end{pmatrix} \tag{5.115}$$

based on N data samples with

$$\mathcal{Y}_N = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_N \end{pmatrix}, \quad \text{and} \quad \Phi_N = \begin{pmatrix} -y_1 & u_1 \\ -y_2 & u_2 \\ \vdots & \vdots \\ -y_{N-1} & u_{N-1} \end{pmatrix} \tag{5.116}$$

Assuming that the model generating data is

$$\mathcal{S}: \quad y_{k+1} = -ay_k + bu_k + w_{k+1} + cw_k, \quad \begin{cases} \mathcal{E}\{u_k^2\} = \sigma_u^2 = \sigma^2 \\ \mathcal{E}\{w_k^2\} = \sigma_w^2 = \sigma^2 \\ \mathcal{E}\{u_j w_k\} = 0, \ \forall i, k \end{cases} \tag{5.117}$$

and that $\{u_k\}$ and $\{w_k\}$ are uncorrelated sequences, we verify that

$$\mathcal{E}\{\frac{1}{N-1}\Phi_N^T\Phi_N\} = \mathcal{E}\{\begin{pmatrix} \frac{1}{N-1}\sum_{k=1}^{N-1} y_k^2 & \frac{-1}{N-1}\sum_{k=1}^{N-1} y_k u_k \\ \frac{-1}{N-1}\sum_{k=1}^{N-1} y_k u_k & \frac{1}{N-1}\sum_{k=1}^{N-1} u_k^2 \end{pmatrix}\} \tag{5.118}$$

Under stationary conditions so that $\mathcal{E}\{y_{k+1}\} = \mathcal{E}\{y_k\}$, $\mathcal{E}\{y_k u_k\} = 0$, $\mathcal{E}\{u_k^2\} = \sigma_u^2$, it holds that

$$\mathcal{E}\{\frac{1}{N-1}\Phi_N^T\Phi_N\} = \begin{pmatrix} \mathcal{E}\{y_k^2\} & 0 \\ 0 & \sigma_u^2 \end{pmatrix} \tag{5.119}$$

and

$$\mathcal{E}\{\frac{1}{N-1}\Phi_N^T\mathcal{Y}_N\} = \mathcal{E}\{\begin{pmatrix} \frac{-1}{N-1}\sum_{k=2}^{N} y_k y_{k-1} \\ \frac{1}{N-1}\sum_{k=2}^{N} y_k u_{k-1} \end{pmatrix}\} \tag{5.120}$$

$$= \begin{pmatrix} a\mathcal{E}\{y_k^2\} - c\sigma_w^2 \\ b\sigma_u^2 \end{pmatrix} \tag{5.121}$$

where the following properties have been used

$$\begin{aligned}
\mathcal{E}\{y_k u_k\} &= 0 && (5.122)\\
\mathcal{E}\{y_{k+1} y_k\} &= \mathcal{E}\{-ay_k^2 + bu_k y_k + w_{k+1}u_k + cw_k u_k\} && (5.123)\\
&= -a\mathcal{E}\{y_k^2\} + b\mathcal{E}\{u_k y_k\} + \mathcal{E}\{w_{k+1}u_k\} + c\mathcal{E}\{w_k u_k\} && (5.124)\\
&= -a\mathcal{E}\{y_k^2\} + c\sigma_w^2 && (5.125)\\
\mathcal{E}\{y_{k+1}^2\} &= \mathcal{E}\{(-ay_k + bu_k + w_{k+1} + cw_k)^2\} && (5.126)\\
&= a^2\mathcal{E}\{y_k^2\} + b^2\mathcal{E}\{u_k^2\} + \mathcal{E}\{w_{k+1}^2\} + c^2\mathcal{E}\{w_k^2\} \\
&- 2ab\mathcal{E}\{y_k u_k\} - 2a\mathcal{E}\{y_k w_{k+1}\} - 2ac\mathcal{E}\{y_k w_k\} \\
&+ 2b\mathcal{E}\{u_k w_{k+1}\} + 2bc\mathcal{E}\{u_k w_k\} + 2\mathcal{E}\{w_{k+1} w_k\} && (5.127)\\
&= a^2\mathcal{E}\{y_k^2\} + b^2\sigma_u^2 + \sigma_w^2 + c^2\sigma_w^2 - 2ac\sigma_w^2 && (5.128)
\end{aligned}$$

Under stationary conditions, we find that $\mathcal{E}\{y_{k+1}^2\} = \mathcal{E}\{y_k^2\}$ so that

$$\mathcal{E}\{y_k^2\} = \frac{\sigma^2}{1-a^2}(b^2 - 2ac + 1 + c^2)$$

For the least-squares estimate based on N samples, it is found that

$$\mathcal{E}\{\widehat{\theta}_N\} = \mathcal{E}\{(\frac{1}{N-1}\Phi_N^T\Phi_N)^{-1}(\frac{1}{N-1}\Phi_N^T\mathcal{Y}_N)\} \tag{5.129}$$

As $N \to \infty$, we have asymptotically

$$\begin{aligned}\widehat{\theta}_\infty &= \lim_{N\to\infty}\mathcal{E}\{\widehat{\theta}_N\} = \lim_{N\to\infty}\mathcal{E}\{(\frac{1}{N-1}\Phi_N^T\Phi_N)^{-1}(\frac{1}{N-1}\Phi_N^T\mathcal{Y}_N)\} \quad (5.130)\\ &= \begin{pmatrix}\mathcal{E}\{y_k^2\} & 0\\ 0 & \sigma^2\end{pmatrix}^{-1}\begin{pmatrix}a\mathcal{E}\{y_k^2\} - c\sigma^2\\ b\sigma^2\end{pmatrix} = \begin{pmatrix}a - c\dfrac{\sigma^2}{\mathcal{E}\{y_k^2\}}\\ b\end{pmatrix}\end{aligned}$$

Thus, we summarize the asymptotical least-squares estimate as $N \to \infty$ as

$$\widehat{\theta}_\infty = \lim_{N\to\infty}\mathcal{E}\{\widehat{\theta}_N\} = \begin{pmatrix}a - c\dfrac{\sigma^2}{\mathcal{E}\{y_k^2\}}\\ b\end{pmatrix} \tag{5.131}$$

The corresponding prediction error is

$$\begin{aligned}\mathcal{E}\{\varepsilon_k^2(\widehat{\theta})\} &= \mathcal{E}\{\frac{1}{N-1}\varepsilon^T(\widehat{\theta})\varepsilon(\widehat{\theta})\} = \mathcal{E}\{\frac{1}{N-1}(\mathcal{Y}_N^T\mathcal{Y}_N - \widehat{\mathcal{Y}}_N^T\widehat{\mathcal{Y}}_N)\} && (5.132)\\ &= \mathcal{E}\{\frac{1}{N-1}\mathcal{Y}_N^T(I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)\mathcal{Y}_N\} && (5.133)\\ &= \mathcal{E}\{\frac{1}{N-1}(\mathcal{Y}_N^T\mathcal{Y}_N)\} - \mathcal{E}\{\widehat{\theta}^T(\frac{1}{N-1}\Phi_N^T\Phi_N)\widehat{\theta}\} && (5.134)\end{aligned}$$

As the number of samples N increases, we have the asymptotic result

$$\begin{aligned}\sigma_\varepsilon^2(\widehat{\theta}_\infty) &= \lim_{N\to\infty}\mathcal{E}\{\varepsilon_k^2(\widehat{\theta}_N)\} && (5.135)\\ &= \lim_{N\to\infty}\mathcal{E}\{(\frac{1}{N-1}\mathcal{Y}_N^T\mathcal{Y}_N - \widehat{\theta}^T(\frac{1}{N-1}\Phi_N^T\Phi_N)\widehat{\theta})\} && (5.136)\\ &= \mathcal{E}\{y_k^2\} - \begin{pmatrix}a - c\sigma^2\mathcal{E}\{y_k^2\}\\ b\end{pmatrix}^T\begin{pmatrix}\mathcal{E}\{y_k^2\} & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}a - c\sigma^2\mathcal{E}\{y_k^2\}\\ b\end{pmatrix}\\ &= (1+c^2)\sigma^2 - \frac{c^2\sigma^4}{\mathcal{E}\{y_k^2\}} && (5.137)\end{aligned}$$

whereas the true parameters give the result

$$\begin{aligned}\mathcal{E}\{\varepsilon_k^2(\theta)\} &= \mathcal{E}\{(\widehat{y}_{k+1|k} - y_{k+1})^2\} = \mathcal{E}\{(-ay_k + bu_k - y_{k+1})^2\} && (5.138)\\ &= \mathcal{E}\{(w_{k+1} + cw_k)^2\} = (1+c^2)\sigma^2 > \mathcal{E}\{\varepsilon_k^2(\widehat{\theta})\} && (5.139)\end{aligned}$$

which proves that the biased least-squares parameter estimates yield a lower prediction-error variance than the true parameters.

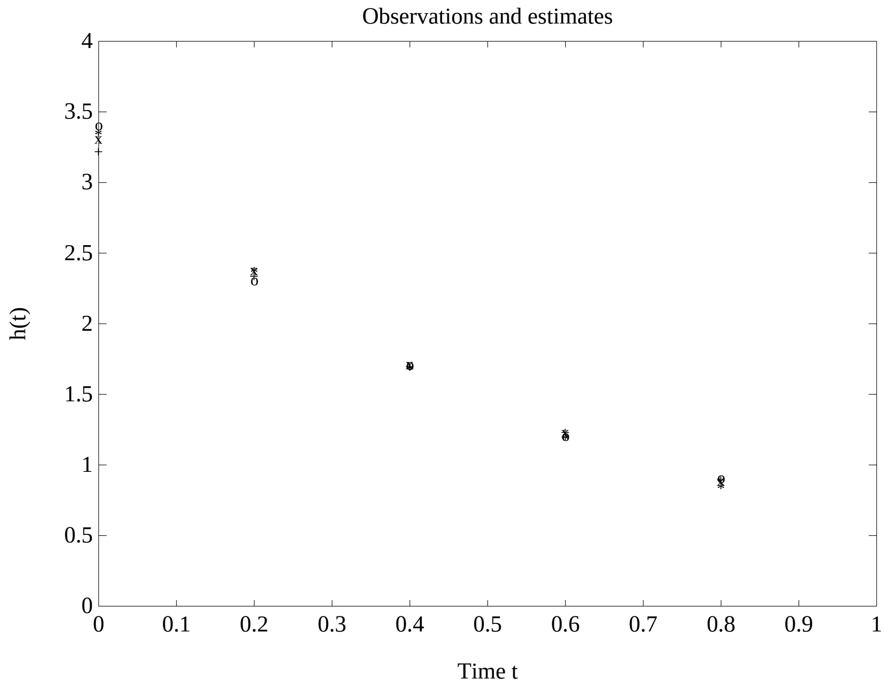

Figure 5.1 Observed data and estimated obtained in Exercise 5.12.

REMARK: The asymptotical prediction error can also be computed as follows

$$\begin{aligned}
\mathcal{E}\{\varepsilon_k^2(\widehat{\theta})\} &= (\widehat{a}-a)^2\mathcal{E}\{y_k^2\} + (\widehat{b}-b)^2\mathcal{E}\{u_k^2\} + 2(\widehat{a}-a)c\mathcal{E}\{y_k w_k\} && (5.140)\\
&= \frac{c^2\sigma^4}{\mathcal{E}\{y_k^2\}} + (1+c^2)\sigma^2 - 2\frac{c^2\sigma^4}{\mathcal{E}\{y_k^2\}} && (5.141)\\
&= (1+c^2)\sigma^2 - \frac{c^2\sigma^4}{\mathcal{E}\{y_k^2\}} && (5.142)
\end{aligned}$$

5.12 Consider the impulse response

$$\mathcal{M}: \quad h(t) = Ke^{-t/\tau}$$

and assume that this is fitted to the given data. As the model is not linear in parameters K, τ one is faced with a number of problems how to estimate the parameters.

a. The least-squares estimate

$$J(K,\tau) = \frac{1}{2}\sum_{k=1}^{N}(h(t_k) - Ke^{-t_k/\tau})^2, \quad N = 5$$

```
function [J]=mim(x);
K=x(1);
tau=x(2);
j=0;
t=[      0;    0.2000;    0.4000;    0.6000;    0.8000];
h=[ 3.4000;    2.3000;    1.7000;    1.2000;    0.9000];
for k=1:5,
pe=h(k)-K*exp(-t(k)/tau);
j=j+pe*pe;
end
J=j/2
end
```

Listing 5.1 A function to minimize in order to solve Exercise 5.12 a.

with the gradient

$$\frac{\partial J}{\partial K} = -\sum_{k=1}^{N} e^{-t_k/\tau}(h(t_k) - K e^{-t_k/\tau}) \tag{5.143}$$

$$\frac{\partial J}{\partial \tau} = -\sum_{k=1}^{N} K \frac{t_k}{\tau^2} e^{-t_k/\tau}(h(t_k) - K e^{-t_k/\tau}) \tag{5.144}$$

Putting the gradient to zero gives the solution (or solutions) to the optimization problem. As this is a difficult problem to solve analytically, we would prefer to solve the problem numerically. A suitable approach is to use an iterative procedure of gradient descent such as

$$\widehat{K}^{(i+1)} = \widehat{K}^{(i)} - \alpha \frac{\partial J}{\partial K}(\widehat{K}^{(i)}, \tau^{(i)}) \tag{5.145}$$

$$\tau^{(i+1)} = \tau^{(i)} - \alpha \frac{\partial J}{\partial \tau}(\widehat{K}^{(i)}, \tau^{(i)}) \tag{5.146}$$

which yields the solution

$$\begin{pmatrix} \widehat{K} \\ \tau \end{pmatrix} = \begin{pmatrix} 3.359 \\ 0.588 \end{pmatrix} \tag{5.147}$$

b. The exponential function formulated in the free variables K and τ is

$$h(t) = K e^{-t/\tau}$$

The relative error of the estimates of K and τ are

$$\frac{\widetilde{K}}{K} = \frac{\widehat{K} - K}{K}, \quad K = h(t)/e^{-t/\tau} \tag{5.148}$$

$$\frac{\widetilde{\tau}}{\tau} = \frac{\widehat{\tau} - \tau}{\tau}, \quad \tau = \frac{t}{\log \dfrac{K}{h(t)}} \tag{5.149}$$

```
function [J]=femb(x);
K=x(1); tau=x(2);
t=[      0;    0.2000;    0.4000;    0.6000;    0.8000];
h=[ 3.4000;    2.3000;    1.7000;    1.2000;    0.9000];
j=0;
for k=1:5,
x1=(tau-t(k)/log(K/h(k)))/tau;
x2=(K-h(k)/exp(-t(k)/tau))/K;
j=j+x1*x1+x2*x2;
end
J=j/2
end
```

Listing 5.2 A function to minimize in order to solve Exercise 5.12 a.

A possible optimization criterion in order to minimize the relative error is

$$J(K,\tau) = \frac{1}{2}\sum_{k=1}^{N} \frac{1}{\tau^2}(\tau - \frac{t_k}{\log K - \log h(t_k)})^2 + \frac{1}{K^2}(K - \frac{h(t_k)}{\exp{(-t_k/\tau)}})^2 \tag{5.150}$$

Numerical minimization of $J(K,\tau)$ yields the result

$$\begin{pmatrix} K \\ \tau \end{pmatrix} = \begin{pmatrix} 3.2081 \\ 0.6200 \end{pmatrix} \tag{5.151}$$

c. A linear regression model can be formulated as

$$\log h(t) = \log K - t\frac{1}{\tau} = \begin{pmatrix} 1 & -t \end{pmatrix} \begin{pmatrix} \log K \\ 1/\tau \end{pmatrix} \tag{5.152}$$

and we find for the data provided

$$\begin{array}{ccccccc} t & = & 0+ & 0.200 & 0.400 & 0.600 & 0.800 \\ h(t) & = & 3.400 & 2.300 & 1.700 & 1.200 & 0.900 \\ \log h(t) & = & 1.2238 & 0.8329 & 0.5306 & 0.1823 & -0.1054 \end{array} \tag{5.153}$$

Least-squares estimation of the parameters

$$\theta = \begin{pmatrix} \log K \\ 1/\tau \end{pmatrix} \tag{5.154}$$

of the linear regression model of Eq. (5.152) yields

$$\widehat{\theta} = \begin{pmatrix} 1.1946 \\ 1.6544 \end{pmatrix} \tag{5.155}$$

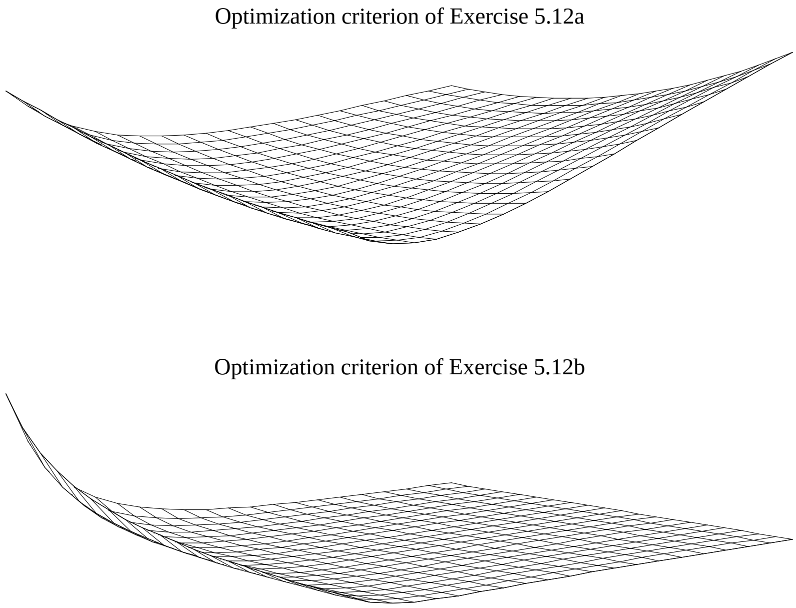

Figure 5.2 Optimization criteria of Exercises 5.12a-b close to the minimum.

with the resultant parameters

$$\begin{pmatrix} K \\ \tau \end{pmatrix} = \begin{pmatrix} 3.3022 \\ 0.6044 \end{pmatrix} \tag{5.156}$$

6

Identification of Time-Series Models

6.1 We consider the data from the system

$$\mathcal{S}: \quad y_k + a y_{k-1} = b u_{k-1} + w_k + c w_{k-1}$$

where $\{u_k\}$ and $\{w_k\}$ are independent zero-mean white noise processes with the variances $\mathcal{E}\{u_k^2\} = \sigma_u^2$ and $\mathcal{E}\{w_k^2\} = \sigma_w^2$, respectively. Looking for the asymptotic parameter estimates of the model

$$\mathcal{M}: \quad y_k + a y_{k-1} = b u_{k-1}$$

it is straightforward to adopt a least-squares estimate based on the linear regression model

$$\mathcal{M}: \quad \mathcal{Y}_N = \Phi_N \theta, \quad \theta = \begin{pmatrix} a \\ b \end{pmatrix} \tag{6.1}$$

with N data samples

$$\mathcal{Y}_N = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_N \end{pmatrix}, \quad \text{and} \quad \Phi_N = \begin{pmatrix} -y_1 & u_1 \\ -y_2 & u_2 \\ \vdots & \vdots \\ -y_{N-1} & u_{N-1} \end{pmatrix} \tag{6.2}$$

and the least-squares solution

$$\widehat{\theta} = \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} = (\Phi_N^T \Phi_N)^{-1} \Phi_N \mathcal{Y}_N \tag{6.3}$$

Assuming that the system generating data is

$$\mathcal{S}: \quad y_{k+1} = -ay_k + bu_k + w_{k+1} + cw_k, \quad \begin{cases} \mathcal{E}\{u_k\} = \sigma_u^2 \\ \mathcal{E}\{w_k\} = \sigma_w^2 \end{cases} \tag{6.4}$$

and that $\{u_k\}$ and $\{w_k\}$ are uncorrelated sequences, we verify that

$$\frac{1}{N-1}\Phi_N^T\Phi_N = \begin{pmatrix} 1N-1\sum_{k=1}^{N-1} y_k^2 & \frac{-1}{N-1}\sum_{k=1}^{N-1} y_k u_k \\ \frac{-1}{N-1}\sum_{k=1}^{N-1} y_k u_k & \frac{1}{N-1}\sum_{k=1}^{N-1} u_k^2 \end{pmatrix} \tag{6.5}$$

and

$$\frac{1}{N-1}\Phi_N^T\mathcal{Y}_N = \begin{pmatrix} \frac{-1}{N-1}\sum_{k=2}^{N} y_k y_{k-1} \\ \frac{1}{N-1}\sum_{k=2}^{N} y_k u_{k-1} \end{pmatrix} \tag{6.6}$$

Under an assumption of ergodicity we find the mathematical expectation $\mathcal{E}\{y_k u_k\} = 0$ and that

$$\begin{aligned}
\mathcal{E}\{y_{k+1}y_k\} &= \mathcal{E}\{-ay_k^2 + bu_k y_k + w_{k+1}u_k + cw_k y_k\} && (6.7)\\
&= -a\mathcal{E}\{y_k^2\} + b\mathcal{E}\{u_k y_k\} + \mathcal{E}\{w_{k+1}y_k\} + c\mathcal{E}\{w_k y_k\} && (6.8)\\
&= -a\mathcal{E}\{y_k^2\} + c\sigma_w^2 && (6.9)\\
\mathcal{E}\{y_{k+1}^2\} &= \mathcal{E}\{(-ay_k + bu_k + w_{k+1} + cw_k)^2\} && (6.10)\\
&= a^2\mathcal{E}\{y_k^2\} + b^2\mathcal{E}\{u_k^2\} + \mathcal{E}\{w_{k+1}^2\} + c^2\mathcal{E}\{w_k^2\} \\
&- 2ab\mathcal{E}\{y_k u_k\} - 2a\mathcal{E}\{y_k w_{k+1}\} - 2ac\mathcal{E}\{y_k w_k\} \\
&+ 2b\mathcal{E}\{u_k w_{k+1}\} + 2bc\mathcal{E}\{u_k w_k\} + 2\mathcal{E}\{w_{k+1}w_k\} && (6.11)\\
&= a^2\mathcal{E}\{y_k^2\} + b^2\sigma_u^2 + \sigma_w^2 + c^2\sigma_w^2 - 2ac\sigma_w^2 && (6.12)
\end{aligned}$$

Under stationary conditions we find that $\mathcal{E}\{y_{k+1}^2\} = \mathcal{E}\{y_k^2\}$ so that

$$\mathcal{E}\{y_k^2\} = \frac{1}{1-a^2}(b^2\sigma_u^2 + (1+c^2-2ac)\sigma_w^2) \tag{6.13}$$

For the special case $\sigma_w^2 = \sigma_u^2$ with signal-to-noise ratio SNR= 1 we have

$$\mathcal{E}\{y_k^2\} = \frac{\sigma^2}{1-a^2}(b^2 - 2ac + 1 + c^2) \tag{6.14}$$

Considering asymptotic properties of Eq. (6.5), we have

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N-1}\Phi_N^T\Phi_N &= \lim_{N\to\infty}\begin{pmatrix} \frac{1}{N-1}\sum_{k=1}^{N-1} y_k^2 & \frac{-1}{N-1}\sum_{k=1}^{N-1} y_k u_k \\ \frac{-1}{N-1}\sum_{k=1}^{N-1} y_k u_k & \frac{1}{N-1}\sum_{k=1}^{N-1} u_k^2 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{E}\{y_k^2\} & \mathcal{E}\{y_k u_k\} \\ \mathcal{E}\{y_k u_k\} & \mathcal{E}\{u_k^2\} \end{pmatrix} = \begin{pmatrix} \mathcal{E}\{y_k^2\} & 0 \\ 0 & \sigma_u^2 \end{pmatrix} && (6.15)
\end{aligned}$$

and for Eq. (6.6)

$$\lim_{N\to\infty} \frac{1}{N-1}\Phi_N^T\mathcal{Y}_N = \lim_{N\to\infty} \begin{pmatrix} \dfrac{-1}{N-1}\sum_{k=2}^N y_k y_{k-1} \\ \dfrac{1}{N-1}\sum_{k=2}^N y_k u_{k-1} \end{pmatrix} \tag{6.16}$$

$$= \begin{pmatrix} \mathcal{E}\{y_{k+1}y_k\} \\ \mathcal{E}\{y_k u_{k-1}\} \end{pmatrix} \tag{6.17}$$

$$= \begin{pmatrix} a\mathcal{E}\{y_k^2\} - c\sigma_w^2 \\ b\sigma_u^2 \end{pmatrix} \tag{6.18}$$

Thus, we summarize the asymptotical least-squares estimate as $N \to \infty$ as

$$\widehat{\theta}_\infty = \lim_{N\to\infty} \widehat{\theta}_N = \lim_{N\to\infty} (\frac{1}{N-1}\Phi_N^T\Phi_N)^{-1}(\frac{1}{N-1}\Phi_N^T\mathcal{Y}_N) \tag{6.19}$$

$$= \begin{pmatrix} \mathcal{E}\{y_k^2\} & 0 \\ 0 & \sigma_u^2 \end{pmatrix}^{-1} \begin{pmatrix} a\mathcal{E}\{y_k^2\} - c\sigma_w^2 \\ b\sigma_u^2 \end{pmatrix} = \begin{pmatrix} a - c\dfrac{\sigma_w^2}{\mathcal{E}\{y_k^2\}} \\ b \end{pmatrix} \tag{6.20}$$

The mean-square prediction error is

$$\frac{1}{N-1}\sum_{k=2}^N \varepsilon_k^2(\widehat{\theta}_N) = \frac{1}{N-1}\mathcal{Y}_N^T(I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)\mathcal{Y}_N \tag{6.21}$$

$$= \frac{1}{N-1}(\mathcal{Y}_N^T\mathcal{Y}_N - \widehat{\mathcal{Y}}_N^T\widehat{\mathcal{Y}}_N) \tag{6.22}$$

$$= \frac{1}{N-1}(\mathcal{Y}_N^T\mathcal{Y}_N - \widehat{\theta}_N^T(\Phi_N^T\Phi_N)\widehat{\theta}_N) \tag{6.23}$$

$$= \frac{1}{N-1}(\widetilde{\theta}_N^T(\Phi_N^T\Phi_N)\widetilde{\theta}_N) \tag{6.24}$$

Using the asymptotical parameter estimate $\widehat{\theta}_\infty$ and Eq. (6.13) we have the mathematical expectation of the mean-square prediction error

$$\mathcal{E}\{\varepsilon_k^2(\widehat{\theta}_\infty)\} = \mathcal{E}\{\lim_{N\to\infty}(\frac{1}{N-1}\mathcal{Y}_N^T\mathcal{Y}_N - \widehat{\theta}_\infty^T(\frac{1}{N-1}\Phi_N^T\Phi_N)\widehat{\theta}_\infty)\} \tag{6.25}$$

$$= \mathcal{E}\{y_k^2\} - \begin{pmatrix} a - c\dfrac{\sigma^2}{\mathcal{E}\{y_k^2\}} \\ b \end{pmatrix}^T \begin{pmatrix} \mathcal{E}\{y_k^2\} & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} a - c\dfrac{\sigma_w^2}{\mathcal{E}\{y_k^2\}} \\ b \end{pmatrix}$$

$$= (1-a^2)\mathcal{E}\{y^2\} + 2ac\sigma_w^2 - \sigma_u^2 b^2 - \frac{c^2\sigma_w^4}{\mathcal{E}\{y_k^2\}} \tag{6.26}$$

$$= (1+c^2)\sigma_w^2 - \frac{c^2\sigma_w^4}{\mathcal{E}\{y_k^2\}} \tag{6.27}$$

whereas the true parameters give the result

$$\begin{aligned}
\mathcal{E}\{\varepsilon_k^2(\theta)\} &= \mathcal{E}\{(\widehat{y}_{k+1} - y_{k+1})^2\} = \mathcal{E}\{(-ay_k + bu_k - y_{k+1})^2\} && (6.28)\\
&= \mathcal{E}\{(w_{k+1} + cw_k)^2\} = (1+c^2)\sigma_w^2 > \mathcal{E}\{\varepsilon_k^2(\widehat{\theta}_\infty)\} && (6.29)
\end{aligned}$$

which proves that the biased least-squares parameters yield a lower expected mean-square prediction error than the true parameters.

Remark: The asymptotical prediction error associated with

$$\widehat{\theta}_\infty = \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} \tag{6.30}$$

may also be computed according to the definition of the prediction error as follows

$$\begin{aligned}
\varepsilon_{k+1} &= \widehat{y}_{k+1} - y_{k+1} && (6.31)\\
&= -(\widehat{a} - a)y_k + (\widehat{b} - b)u_k - w_{k+1} - cw_k && (6.32)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}\{\varepsilon_k^2(\widehat{\theta})\} &= (\widehat{a} - a)^2\mathcal{E}\{y_k^2\} + (\widehat{b} - b)^2\mathcal{E}\{u_k^2\} + \mathcal{E}\{(w_{k+1} + cw_k)^2\} && (6.33)\\
&+ 2(\widehat{a} - a)\mathcal{E}\{y_k(w_{k+1} + cw_k)\} - 2(\widehat{b} - b)\mathcal{E}\{u_k(w_{k+1} + cw_k)\}\\
&- 2(\widehat{a} - a)(\widehat{b} - b)\mathcal{E}\{y_k u_k\}\\
&= (\widehat{a} - a)^2\mathcal{E}\{y_k^2\} + (\widehat{b} - b)^2\sigma_u^2 && (6.34)\\
&+ (1+c^2)\sigma_w^2 + 2(\widehat{a} - a)c\mathcal{E}\{y_k w_k\}\\
&= \frac{c^2\sigma_w^4}{\mathcal{E}\{y_k^2\}} + (1+c^2)\sigma_w^2 - 2\frac{c_w^2\sigma^4}{\mathcal{E}\{y_k^2\}} && (6.35)\\
&= (1+c^2)\sigma_w^2 - \frac{c^2\sigma_w^4}{\mathcal{E}\{y_k^2\}} && (6.36)
\end{aligned}$$

6.2 Notice that

$$\widehat{x} = -Q_2^{-1}q_1$$

minimizes the function

$$V(x) = \frac{1}{2}x^T Q_2 x + x^T q_1 + q_0$$

Now introduce the error

$$e^{(i)} = x^{(i)} - \widehat{x} = x^{(i)} + Q_2^{-1}q_1$$

and consider the error norm

$$\|e^{(i)}\|_P^2 = (e^{(i)})^T P e^{(i)}$$

The optimization algorithm is supposed to be

$$x^{(i+1)} = x^{(i)} - \alpha_i (V''(x^{(i)}))^{-1} V'(x^{(i)})$$

The error norm develops in one iteration of the

$$\begin{aligned} e^{(i+1)} &= e^{(i)} - \alpha_i (V''(x^{(i)}))^{-1} V'(x^{(i)}) && (6.37)\\ &= e^{(i)} - \alpha_i Q_2^{-1}(Q_2 x^{(i)} + q_1) = (1-\alpha_i) e^{(i)} && (6.38) \end{aligned}$$

The error norm develops in one iteration of the as

$$\|e^{(i+1)}\|_P^2 = (1-\alpha_i)^2 \|e^{(i)}\|_P^2$$

so that $\|e^{(i)}\|_P^2$ decreases for step lengths chosen in the range $0 < \alpha_i < 2$.

6.3 Consider data generated by the MA-process

$$\mathcal{S}: \quad y_k = b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_m u_{k-m} + v_k$$

where $\{v_k\}$ is a colored noise sequence. Assume that the following model is adopted

$$\mathcal{M}: \quad y_k = b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_m u_{k-m}$$

Least-squares identification based on the linear regression model

$$\mathcal{M}': \quad y_k = \phi_k^T \theta = \begin{pmatrix} u_{k-1} & u_{k-2} & \dots & u_{k-m} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \qquad (6.39)$$

or $\mathcal{Y}_N = \Phi_N \theta$ with

$$\Phi_n = \begin{pmatrix} u_m & u_{m-1} & \dots & u_1 \\ u_{m+1} & u_m & \dots & u_2 \\ \vdots & \vdots & & \vdots \\ u_{N-1} & u_{N-2} & \dots & u_{N-m} \end{pmatrix}, \quad \text{and} \quad \mathcal{Y}_N = \begin{pmatrix} y_{m+1} \\ y_{m+2} \\ \vdots \\ y_N \end{pmatrix} \qquad (6.40)$$

The least-squares estimate is

$$\begin{aligned} \widehat{\theta} &= (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{Y}_N = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T (\Phi_N \theta + v) && (6.41)\\ &= \theta + (\Phi_N^T \Phi_N)^{-1} \Phi_N^T v && (6.42) \end{aligned}$$

with the expected value

$$\begin{aligned} \mathcal{E}\{\widehat{\theta}\} &= \mathcal{E}\{(\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{Y}_N\} = \mathcal{E}\{(\Phi_N^T \Phi_N)^{-1} \Phi_N^T (\Phi_N \theta + v)\} && (6.43)\\ &= \theta + \mathcal{E}\{(\frac{1}{N-m} \Phi_N^T \Phi_N)^{-1} (\frac{1}{N-m} \Phi_N^T v)\} && (6.44) \end{aligned}$$

As Φ_N depends only on past values $\{u_k\}$, we have

$$\lim_{N\to\infty} \mathcal{E}\{(\frac{1}{N-m}\Phi_N^T\Phi_N)^{-1}(\frac{1}{N-m}\Phi_N^T v)\} = 0$$

if and only if $\mathcal{E}\{\Phi_N^T v\} = 0$, *i.e.*, if and only if $\{v_k\}$ is uncorrelated with $\{u_k\}$. According to Def. B9 in Appendix B of *System Modeling and Identification* (p. 422) we conclude that such an estimate is consistent (in probability) with the probability limit

$$\text{plim } \widehat{\theta} = \theta$$

As a result, we are not faced with the dilemma encountered in the context of least-squares identification of autoregressive moving average models in which the the noise sequence $\{v_k\}$ and the output sequence $\{y_k\}$ are generally correlated. The only essential restriction imposed for MA-models is the the input sequence $\{u_k\}$ may not be generated by output feedback which would introduce undesired correlation.

6.4 Assume that the noise components $\{v_k\}_{k=1}^N$ are arranged as the vector v with the associated probability density function

$$f(v) = \prod_{k=1}^N f(v_k) = \prod_{k=1}^N \frac{v_k}{\sigma^2} e^{-\dfrac{v_i^2}{2\sigma^2}} = \frac{1}{\sigma^{2N}}(\prod_{k=1}^N v_k) e^{-\dfrac{v^T v}{2\sigma^2}}, \quad \forall v_k \geq 0$$

Assuming a linear model

$$y_k = \phi_k^T \theta + v_k \tag{6.45}$$

$$\varepsilon_k = y_k - \phi_k^T \bar{\theta} \tag{6.46}$$

and given the observations $\mathcal{Y}_N$ and the regressor matrix Φ_N thus results in the following likelihood function for θ

$$L(\bar{\theta}, \sigma^2) = (\prod_{k=1}^N f(\varepsilon_k | \bar{\theta}, \sigma^2)) \tag{6.47}$$

$$= \frac{1}{\sigma^{2N}}(\prod_{k=1}^N \varepsilon_k) \exp\big((-\frac{1}{2\sigma^2})(\mathcal{Y}_N - \Phi_N\bar{\theta})^T(\mathcal{Y}_N - \Phi_N\bar{\theta})\big) \tag{6.48}$$

for $\varepsilon_k \geq 0$, $\forall k$ and the log-likelihood function, $\forall \varepsilon_k \geq 0$, is

$$\log L(\bar{\theta}, \sigma^2) = -N \log \sigma^2 - \sum_{k=1}^N \log \varepsilon_k - \frac{1}{2\sigma^2}(\mathcal{Y}_N - \Phi_N\bar{\theta})^T(\mathcal{Y}_N - \Phi_N\bar{\theta})$$

The partial derivatives with respect to σ^2 and $\bar{\theta}$ are

$$\frac{\partial L(\bar{\theta})}{\partial \bar{\theta}} = -\sum_{k=1}^{N} \frac{1}{y_k - \phi_k \bar{\theta}} \phi_k - \frac{1}{\sigma^2}(-\Phi_N^T \mathcal{Y} + \Phi_N^T \Phi_N \bar{\theta}) \tag{6.49}$$

$$\frac{\partial L(\bar{\theta})}{\partial \sigma^2} = -N\sigma^2 + \frac{1}{2\sigma^4}(\mathcal{Y}_N - \Phi_N \bar{\theta})^T (\mathcal{Y}_N - \Phi_N \bar{\theta}) \tag{6.50}$$

Putting the gradient to zero shows that there are extrema for

$$\sigma^2 = \frac{1}{2N}(\mathcal{Y}_N - \Phi_N \bar{\theta})^T (\mathcal{Y}_N - \Phi_N \bar{\theta})$$

If we substitute σ^2 in the log-likelihood function, then we obtain

$$\begin{aligned}\log L(\bar{\theta}) &= -N \log(\frac{1}{2N}(\mathcal{Y}_N - \Phi_N \bar{\theta})^T (\mathcal{Y}_N - \Phi_N \bar{\theta})) \\ &- \frac{1}{2}\sum_{k=1}^{N} \log \varepsilon_k^2 - N - N\log(\sum_{k=1}^{N} \varepsilon_k^2) - \frac{1}{2}\sum_{k=1}^{N} \log \varepsilon_k^2 \end{aligned} \tag{6.51}$$

and for its gradient

$$\frac{\partial L(\bar{\theta})}{\partial \bar{\theta}} = -\sum_{k=1}^{N} \frac{1}{\varepsilon(\bar{\theta})} \phi_k - 2N \frac{\Phi_N \varepsilon(\bar{\theta})}{\varepsilon(\bar{\theta})^T \varepsilon(\bar{\theta})}$$

which can be solved numerically by evaluating $L(\bar{\theta})$ and $\partial L(\bar{\theta})/\partial \bar{\theta}$, see Appendic C of *System Modeling and Identification*.

6.5 The system generating data is assumed to be

$$y_{k+1} = a(u_k + w_k), \quad w_k \in \mathcal{N}(0, \sigma^2), \quad \mathcal{E}\{w_i w_j\} = \sigma^2 \delta_{ij}$$

Considering this as a linear regression problem we have

$$\mathcal{M}: \quad y_{k+1} = a u_k$$

with the observation and regressor matrices

$$\mathcal{Y}_N = \begin{pmatrix} y_1 & y_2 & \dots & y_N \end{pmatrix}^T \tag{6.52}$$

and

$$\Phi_N = \begin{pmatrix} u_0 & u_1 & \dots & u_{N-1} \end{pmatrix}^T \tag{6.53}$$

Evaluate

$$\Phi_N^T \Phi_N = \sum_{k=1}^{N-1} u_k^2, \quad \text{and} \quad \Phi_N^T \mathcal{Y}_N = \sum_{k=1}^{N} y_k u_{k-1}$$

and

$$\widehat{\theta}_N = (\frac{1}{N}\Phi_N^T\Phi_N)^{-1}(\frac{1}{N}\Phi_N^T\mathcal{Y}_N)$$

with the asymptotical result

$$\lim_{N\to\infty}\widehat{\theta}_N = (\mathcal{E}\{u_k^2\})^{-1}\mathcal{E}\{y_k u_{k-1}\} = C_{uu}^{-1}C_{yu}(-1)$$

where

$$C_{yu}(-1) = \mathcal{E}\{(au_k + aw_k)u_k\} = aC_{uu}(0) + aC_{wu}(0)$$

When $C_{wu}(0) = 0$ we find that the parameter estimate is consistent as

$$\lim_{N\to\infty}\widehat{\theta}_N = C_{uu}^{-1}(0)(aC_{uu}(0) + aC_{wu}(0)) = a + aC_{uu}^{-1}(0)C_{wu}(0) = a$$

An alternative solution to this problem in the case of a known value σ^2 is to adopt the maximum-likelihood approach with the log-likelihood function

$$\log L(\theta) = -N\log\sqrt{2\pi} - \frac{1}{2}N\log\theta^2\sigma^2 - \frac{1}{2\theta^2\sigma^2}\sum_{k=1}^{N}(y_k - \phi_k\theta)^2, \quad \theta = a$$

with the derivative

$$\frac{\partial\log L(\theta)}{\partial\theta} = -N\frac{1}{\theta} + \frac{1}{\theta^3\sigma^2}\sum_{k=1}^{N}(y_k - \phi_k\theta)^2 + \frac{1}{\theta^2\sigma^2}\sum_{k=1}^{N}\phi_k(y_k - \phi_k^T\theta)$$

As it is a difficult task to analytically solve for the optimum of the log-likelihood function, it is common practice to solve such problems numerically, see Fig. 6.1. Unfortunately, few software packages support the numerical solution of such problems.

6.6 The idea behind the instrumental variable method is to find a set of variables Z for the linear regression model $\mathcal{Y}_N = \Phi_N\theta + v$ or

$$\varepsilon = v = \mathcal{Y}_N - \Phi_N\theta$$

so that

$$0 = Z^T v = Z^T\mathcal{Y}_N - Z^T\Phi_N\theta = Z^T\varepsilon$$

which enables the solution

$$\widehat{\theta} = (Z^T\Phi_N)^{-1}Z^T\mathcal{Y}_N$$

Thus we have the requested augmented system equation

$$\begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}\begin{pmatrix} \varepsilon \\ \widehat{\theta} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_N \\ 0 \end{pmatrix} \tag{6.54}$$

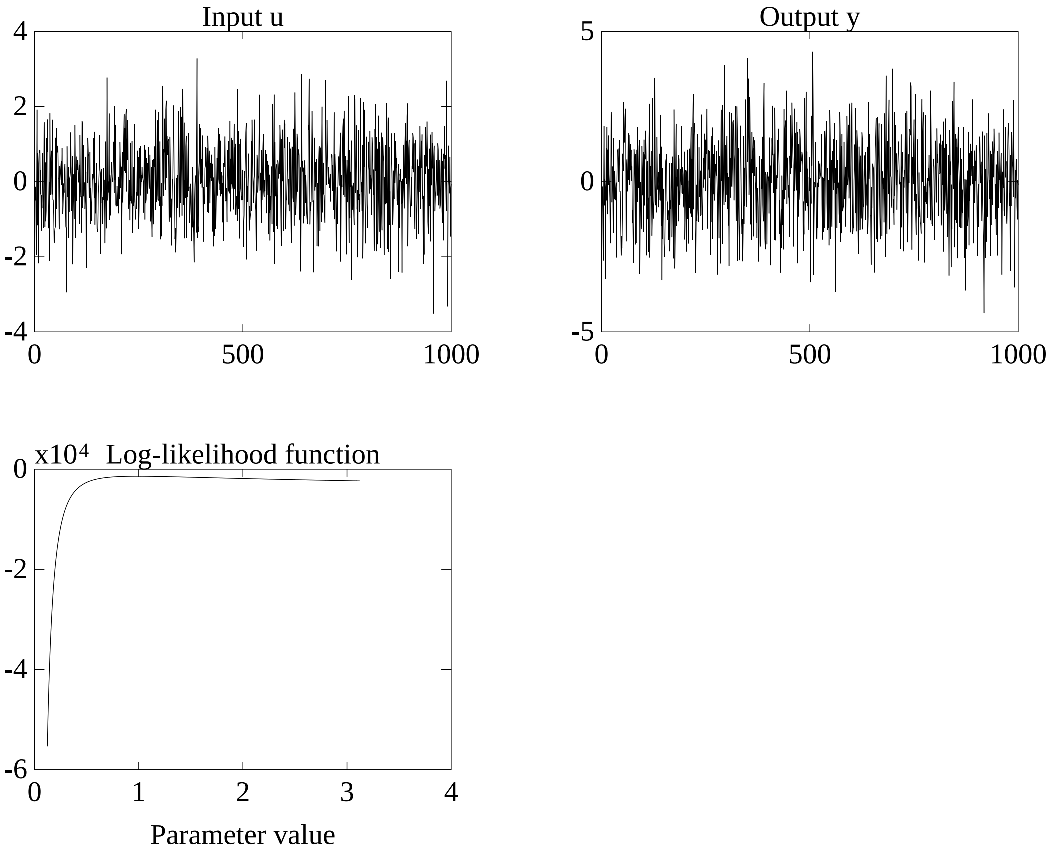

Figure 6.1 Log-likelihood function for the example in Exercise 6.5 ($a = 1$) based on $N = 1000$ data points with minimum at $\bar{\theta} = 0.987$.

6.7 The augmented system equation gives

$$\begin{pmatrix} \varepsilon \\ \widehat{\theta} \end{pmatrix} 0 \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{Y}_N \\ 0 \end{pmatrix} \tag{6.55}$$

and it is straightforward to verify that

$$\begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I - \Phi_N(Z^T\Phi_N)^{-1}Z^T & \Phi_N(Z^T\Phi_N)^{-1} \\ (Z^T\Phi_N)^{-1}Z^T & -(Z^T\Phi_N)^{-1} \end{pmatrix} \tag{6.56}$$

so that

$$\begin{pmatrix} \varepsilon \\ \widehat{\theta} \end{pmatrix} \begin{pmatrix} I - \Phi_N(Z^T\Phi_N)^{-1}Z^T & \Phi_N(Z^T\Phi_N)^{-1} \\ (Z^T\Phi_N)^{-1}Z^T & -(Z^T\Phi_N)^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{Y}_{\mathcal{N}} \\ 0 \end{pmatrix} \tag{6.57}$$

6.8 We consider a linear regression model $\mathcal{Y}_N = \Phi_N\theta + v$ with $\mathrm{Cov}(v) = \Sigma_v$ and $\mathcal{E}\{v\} = 0$. The augmented system equation for the instrumental variable method is

$$\begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \widehat{\theta} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_N \\ 0 \end{pmatrix} \tag{6.58}$$

By subtracting

$$\begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} \Phi_N\theta \\ 0 \end{pmatrix} \tag{6.59}$$

from Eq. (6.58) we obtain the relationship

$$\begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \widetilde{\theta} \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} \tag{6.60}$$

Hence we have the requested relationship

$$\mathrm{Cov}\{ \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \widetilde{\theta} \end{pmatrix} = \mathrm{Cov}\{ \begin{pmatrix} v \\ 0_{p\times 1} \end{pmatrix} = \begin{pmatrix} \Sigma_v & 0_{N\times p} \\ 0_{p\times N} & 0_{p\times p} \end{pmatrix} \tag{6.61}$$

or

$$\begin{aligned}
\mathrm{Cov}\{ \begin{pmatrix} \varepsilon \\ \widetilde{\theta} \end{pmatrix} \} &= \mathrm{Cov} \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix} \\
&= \mathrm{Cov}\{ \begin{pmatrix} v \\ 0_{p\times 1} \end{pmatrix} && (6.62) \\
&\approx \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-1} \mathrm{Cov}\{ \begin{pmatrix} v \\ 0_{p\times 1} \end{pmatrix} \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-T} && (6.63) \\
&= \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_v & 0_{N\times p} \\ 0_{p\times N} & 0_{p\times p} \end{pmatrix} \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-T} && (6.64)
\end{aligned}$$

By applying the result of Exercizes 6.7 to Eq. (6.64) we obtain the covariance estimate

$$\begin{aligned}
\mathrm{Cov}\{ \begin{pmatrix} \varepsilon \\ \widetilde{\theta} \end{pmatrix} &\approx \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_v & 0_{N\times p} \\ 0_{p\times N} & 0_{p\times p} \end{pmatrix} \begin{pmatrix} I & \Phi_N \\ Z^T & 0 \end{pmatrix}^{-T} && (6.65) \\
&= \begin{pmatrix} P\Sigma_v P^T & P\Sigma_v R^T \\ R\Sigma_v P^T & R\Sigma_v R^T \end{pmatrix} && (6.66)
\end{aligned}$$

for $P = (I - \Phi_N(Z^T\Phi_N)^{-1}Z^T)$ and $R = (Z^T\Phi_N)^{-1}Z^T$.

6.9 Consider the functions

$$\begin{aligned}
f_1(A) &= \log\det A && (6.67) \\
f_2(A) &= \mathrm{tr}(WA) && (6.68)
\end{aligned}$$

where W is a symmetric positive definite weighting matrix.

For square symmetric matrices A and A_0 where $A \geq A_0 > 0$ there is a matrix factorization such that

$$A_0 = L^T L$$

Thus for any matrix $\Delta = \Delta^T > 0$ we can evaluate the function

$$\begin{aligned} f_1(A_0 + \Delta) - f_1(A_0) &= \log\det(A_0 + \Delta) - \log\det(A_0) && (6.69)\\ &= \log\det(I + L^{-T}\Delta L^{-1}) && (6.70)\\ &= \log\det T^{-1}(I + L^{-T}\Delta L^{-1})T && (6.71)\\ &= \log\det(I + T^{-1}L^{-T}\Delta L^{-1}T) && (6.72) \end{aligned}$$

where T is any invertible matrix. In particular, we can choose T according to a similarity transformation such that

$$T^{-1}L^{-T}\Delta L^{-1}T = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_N \end{pmatrix} \tag{6.73}$$

so that we can verify the inequality

$$f_1(A_0 + \Delta) - f_1(A_0) = \sum_{k=1}^{n} \log(1 + \lambda_k) > 0$$

An alternative solution is

$$\begin{aligned} f_1(A) - f_1(A_0) &= \log\det A - \log\det A_0 && (6.74)\\ &= \log\prod_{k=1}^{n}\lambda_k(A) - \log\prod_{k=1}^{n}\lambda_k(A_0) && (6.75)\\ &= \prod_{k=1}^{n}(\log\lambda_k(A) - \log\lambda_k(A_0)) \geq 0 && (6.76) \end{aligned}$$

where $\lambda_k(A); \quad k = 1, 2, \dots, n$ denotes the kth eigenvalue of the $n \times n$—matrix A.

For the second inequality in Eq. (6.68) with a positive definite weighting matrix W factorized as $W = L^T L$, we find that

$$f_2(A_0 + \Delta) - f_2(A_0) = \mathrm{tr}(W\Delta) = \mathrm{tr}(L\Delta L^T) \geq 0$$

For square symmetric matrices A and A_0 where $A > A_0$ it holds that $f_1(A) \geq f_1(A_0)$ and $f_2(A) \geq f_2(A_0)$.

6.10 The sequence $\{v_k\}$ is a sequence of independent identically distributed stochastic variables, each with the probability density function $f(x) = \mu e^{-\mu x}$. We have the following equations for the system and the model

$$\begin{aligned} \mathcal{S}: \quad v_k &= y_k + a y_{k-1} - b u_{k-1} && (6.77) \\ \mathcal{M}: \quad \varepsilon_k &= y_k + \widehat{a} y_{k-1} - \widehat{b} u_{k-1} && (6.78) \end{aligned}$$

An assumption underlying the formulation of the likelihood function is that for true parameter *theta* we have

$$\begin{aligned} \varepsilon_k(\theta) &= y_k + a y_{k-1} - b u_{k-1} = v_k && (6.79) \\ f(\varepsilon_k(\theta)) &= f(v_k) && (6.80) \end{aligned}$$

The likelihood function is then

$$\begin{aligned} L(\bar{\theta}, \mu) &= \prod_{k=1}^{N} f(\varepsilon_k) && (6.81) \\ &= \prod_{k=1}^{N} \mu \exp\big(-\mu(y_k + \widehat{a} y_{k-1} - \widehat{b} u_{k-1})\big) && (6.82) \end{aligned}$$

and the log-likelihood function is

$$\log L(\bar{\theta}, \mu) = N \log \mu - \mu \sum_{k=1}^{N} \big(y_k + \widehat{a} y_{k-1} - \widehat{b} u_{k-1}\big), \quad \bar{\theta} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad (6.83)$$

The partial derivative with respect to μ is

$$\frac{\partial \log L(\bar{\theta}, \mu)}{\partial \mu} = \frac{N}{\mu} - \sum_{k=1}^{N} \big(y_k + \widehat{a} y_{k-1} - \widehat{b} u_{k-1}\big)$$

which suggest μ to be chosen as

$$\widehat{\mu} = \frac{N}{\displaystyle\sum_{k=1}^{N} \big(y_k + \widehat{a} y_{k-1} - \widehat{b} u_{k-1}\big)}$$

If μ is substituted for $\widehat{\mu}$ in the log-likelihood function, we have

$$\log L(\bar{\theta}) = N \log N - N - \log \sum_{k=1}^{N} \big(y_k + \widehat{a} y_{k-1} - \widehat{b} u_{k-1}\big)$$

This modified optimization problem may be approached by numerical methods in order to find the optimal $\bar{\theta}$.

7

Modeling

7.1 XSAssuming $y(t)$ to be the observed variable, we solve the differential equation given in the formulation of this exercise in the textbook

$$
\begin{aligned}
x(t) &= e^{-r(t-t_0)}x(t_0) && (7.1)\\
y(t) &= ce^{-r(t-t_0)}x(t_0) = e^{-r(t-t_0)}y(t_0) && (7.2)
\end{aligned}
$$

Taking logarithms we obtain

$$\log y(t) = \log y(t_0) - r(t - t_0)$$

and we can solve for the desired transfer coefficient by means of the equation

$$r = \frac{\log y(t_0) - \log y(t)}{t - t_0}$$

7.2 Consider the serum and gastrointestinal compartments with the distribution volumes V_1 and V_2 and the concentration x_1 and x_2, respectively.

$$
\begin{aligned}
\dot{x}_1 &= -r_1x_1 + r_2x_2 \quad + r_3x_3 - r_4x_1 && (7.3)\\
\dot{x}_2 &= -r_2x_2 && (7.4)\\
\dot{x}_3 &= -r_3x_3 + r_4x_1 && (7.5)
\end{aligned}
$$

where oral intake loads the gastrointestinal compartment x_2 whereas intravenous (i.v.) infusion loads the serum compartment x_1. Visual interpretation of the graphs of data suggests that a first order model might suffice for intravenous (i.v.) intake, *i.e.*, a first hypothesis is that the tissue compartment might be neglected. A reduced set of equation to model this case is

$$
\begin{aligned}
\dot{x}_1 &= -r_1x_1 + r_2x_2 && (7.6)\\
\dot{x}_2 &= -r_2x_2 && (7.7)
\end{aligned}
$$

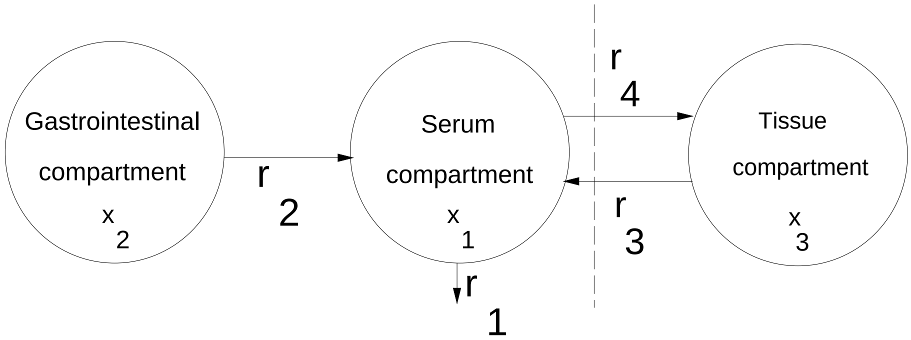

Figure 7.1 Compartment model of Exercise 7.2.

Loading of the serum compartment at time $t = t_0$ gives rise to elimination according to the equation

$$x_1(t) = e^{-r_1(t-t_0)} x_{10}, \quad \text{where} \quad x_{10} = \frac{m_1}{V_1}$$

and where

$$\begin{cases} m_i, & \text{Dose loaded into compartment } i \\ V_i, & \text{Distribution volume of compartment } i \end{cases} \quad i = 1,2,3 \tag{7.8}$$

The transfer coefficient r_1 and the distribution volume can thus be determined by fitting data to a model.

Numerical optimization can be done of the following least-squares criterion

$$J_1(K, r_1) = \sum_{k=1}^{N} (x_1(t_k) - K e^{-r_1 t_k})^2; \quad K = \frac{m}{V} \tag{7.9}$$

which is nonlinear in parameters. Another approach is to optimize the following least-squares criterion model

$$\begin{aligned} J_2(K, r_1) &= \sum_{k=1}^{N} (\log x_1(t_k) - \log K + r_1 t_k)^2 && (7.10) \\ &= \sum_{k=1}^{N} (\log x_1(t_k) - \log m + \log V + r_1 t_k)^2 && (7.11) \end{aligned}$$

based on the linear regression

$$\mathcal{M} : \quad \mathcal{Y}_N = \Phi_N \theta$$

which can be organized as

$$\mathcal{Y}_N = \begin{pmatrix} \log x_1(t_1) - \log m \\ \log x_1(t_2) - \log m \\ \vdots \\ \log x_1(t_N) - \log m \end{pmatrix}, \quad \text{and} \quad \Phi_N = \begin{pmatrix} -1 & -t_1 \\ -1 & -t_2 \\ \vdots & \vdots \\ -1 & -t_N \end{pmatrix} \tag{7.12}$$

and the parameter vector

$$\theta = \begin{pmatrix} \log K \\ r_1 \end{pmatrix} \tag{7.13}$$

The advantage of optimizing Eq. (7.12) as compared to Eq. (7.9) is that it is linear in the parameters $\log V$ and r_1 and can be solved as an ordinary least-squares problem. Application to the data provided in the textbook yields

$$\begin{pmatrix} \log V \\ r_1 \end{pmatrix} = \begin{pmatrix} 3.2754 \\ 0.2387 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} V \\ r_1 \end{pmatrix} = \begin{pmatrix} 56.64\ [l] \\ 0.2572\ [h] \end{pmatrix} \tag{7.14}$$

The distribution volume of 56 [l] might appear large in comparison to the human blood volume of about 5 [l]. Standard interpretations of such results are that the drug is somehow chemically bound to some component of the blood and released slowly. Another interpretation is that the blood compartment and the tissue compartment are indistinguishable from the point of view of drug distribution.

When loading the gastrointestinal compartment we have

$$\dot{x}_1 = -r_1 x_1 + r_2 x_2 \tag{7.15}$$

$$\dot{x}_2 = -r_2 x_2 \qquad \begin{cases} x_1(t) = \dfrac{r_2 x_{20}}{r_2 - r_1}(e^{-r_2 t} - e^{-r_1 t}) \\ x_2(t) = x_{20} e^{-r_2 t} \end{cases} \tag{7.16}$$

By minimizing the function (for instance by means of the procedure “FMINS” in Matlab)

$$J_3(K, r_1, r_2) = \sum_{k=1}^{N} (x_1(t_k) - K(e^{-r_2 t_k} - e^{-r_1 t_k}))^2$$

we obtain the numerical values

$$\begin{pmatrix} K \\ r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 4.22 \cdot 10^{-3} \\ 0.249 \\ 2.38 \end{pmatrix} \tag{7.17}$$

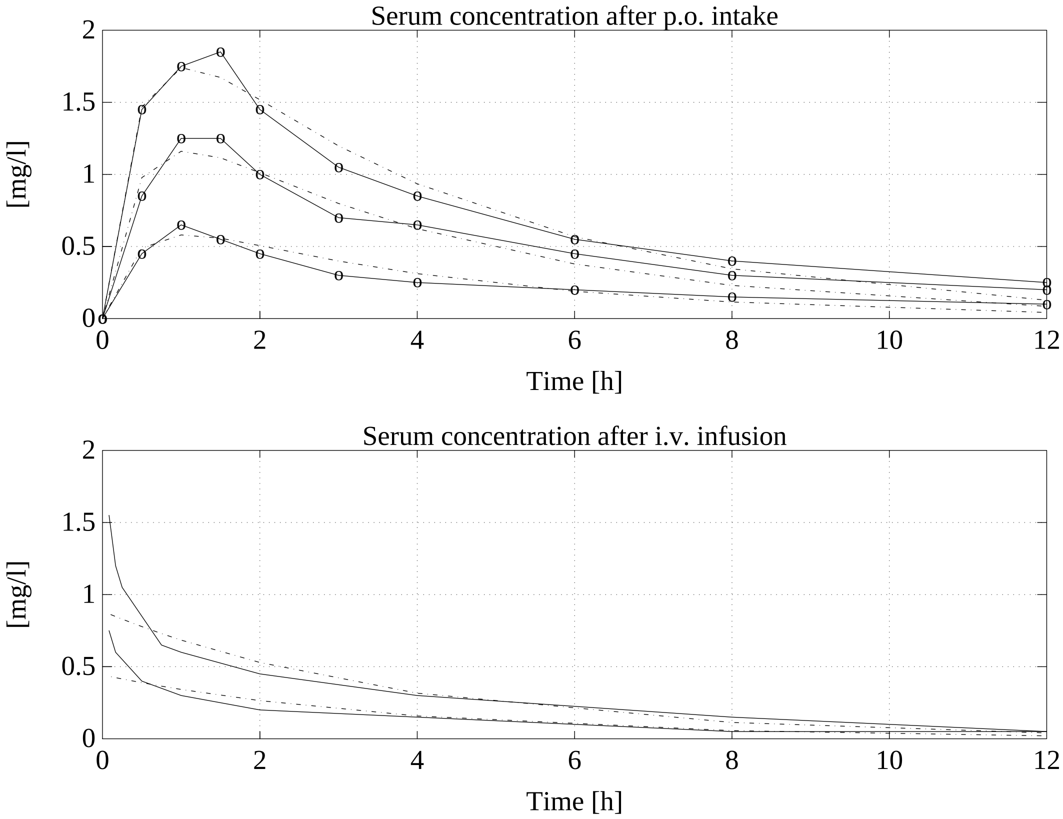

Figure 7.2 Fitting of two-compartment model to the data in Exercise 7.2

The fitted two-compartment model output and data are shown in Fig. 7.17. The estimate of the transfer coefficient r_1 is consistent with the estimate provided in Eq. (7.14). The transfer coefficient r_2 representing the absorption of the drug from the gastrointestinal tract is about ten times larger than the time constant r_1 representing elimination. Further refinement can be achieved by additional modeling of the tissue compartment.

A major drawback with the explicit method presented above is that the complexity of the optimization problem increases at a fast rate as the number of compartments increases. An alternative means of analysis can be approached by considering the methods of Chapter 12 of *System Modeling and Identification*. If we model the loading of a certain compartment by means of the linear model

$$\dot{x}(t) = Ax(t) + B\delta(t) \tag{7.18}$$

$$sX(s) = AX(s) + B \tag{7.19}$$

then we can apply the operator transform

$$\lambda = \frac{1}{1+\tau s}$$

to Eq. (7.19) we find

$$\frac{1}{\tau}\frac{1-\lambda(s)}{\lambda(s)}X = AX + B$$

which yields the linear regression model

$$X = (I + \tau A)(\lambda(s)X) + \tau(\lambda(s)B)$$

which is expressed in the compartment states x and the input.

Application of the same methodology to a transfer function model from input to (a restricted set of) outputs according to the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\delta(t) && (7.20)\\ y &= Cx \quad \Rightarrow Y(s) = C(sI-A)^{-1}B && (7.21) \end{aligned}$$

For instance, for the two-compartment model given above with $y(t)$ being the elimination rate $r_1x_1(t)$ and with loading of the gastrointestinal compartment at the time $t = 0$, we arrive at the transfer function model

$$Y(s) = \frac{r_1r_2}{(s+r_1)(s+r_2)}\mathcal{L}\{\frac{m_2}{V_2}\delta(t)\}$$

Application of the operator transform

$$\lambda = \frac{1}{1+\tau s}$$

gives the linear regression model

$$\begin{aligned} [(1-\lambda)^2Y(s)] &= -(r_1+r_2)[\tau\lambda(1-\lambda)Y(s)] - r_1r_2[\tau^2\lambda^2Y(s)] && (7.22)\\ &+ r_1r_2\frac{m_2}{V_2}[\tau^2\lambda^2\mathcal{L}\{\delta(t)\}] && (7.23) \end{aligned}$$

or in the time domain

$$\begin{aligned} [(1-\lambda)^2\{y(t)\}] &= -(r_1+r_2)[\tau\lambda(1-\lambda)\{y(t)\}] - r_1r_2[\tau^2\lambda^2\{y(t)\}] && (7.24)\\ &+ r_1r_2\frac{m_2}{V_2}[\tau^2\lambda^2\{\delta(t)\}] && (7.25) \end{aligned}$$

which can be solved by linear regression methods with results similar to Eq. (7.14) being obtained. The continuous-time modeling exhibits nice properties as the model-order dependent complexity effectively precludes explicit criterion minimization for model complexity greater than model order two or three.

7.3 We consider the logistic curve

$$y(u) = x(u) + \epsilon = \frac{1}{1 + \exp(-(\alpha + \beta u))} + \epsilon$$

By making the suggested transformation

$$\begin{aligned}
z &= \log \frac{y}{1-y} && (7.26)\\
&= \log \frac{1 + \epsilon(1 + \exp(-(\alpha + \beta u)))}{\exp(-(\alpha + \beta u)) - \epsilon(1 + \exp(-(\alpha + \beta u)))} && (7.27)\\
&= \log(1 + \epsilon(1 + \exp(-(\alpha + \beta u)))) - \log \exp(-(\alpha + \beta u)) && (7.28)\\
&- \log(1 - \frac{\epsilon(1 + \exp(-(\alpha + \beta u)))}{\exp(-(\alpha + \beta u))}) && (7.29)\\
&= (\alpha + \beta u) \log(1 + \epsilon(1 + \exp(-(\alpha + \beta u)))) && (7.30)\\
&- \log(1 - \frac{\epsilon(1 + \exp(-(\alpha + \beta u)))}{\exp(-(\alpha + \beta u))}) && (7.31)
\end{aligned}$$

Using the standard identity $\log(1 + x) \leq x$ we obtain for small ϵ that

$$z \approx (\alpha + \beta u) + \epsilon \frac{(1 + \exp(-(\alpha + \beta u)))^2}{\exp(-(\alpha + \beta u))}$$

Assuming observations $\{\epsilon_k\}$ of ϵ to be small and statistically independent, we would expect such a linear regression model to yield unbiased estimates. The magnification factor is, however, very large for certain values of u.

7.4 In this exercise is considered a cylindrical water tank with cross section area A and outlet area a

a. The differential equation

$$A\frac{dh}{dt} = -a\sqrt{2gh}$$

is solved by separation of variables, resulting in

$$\frac{h'}{2\sqrt{h}} = -\frac{a}{A}\sqrt{\frac{g}{2}}, \quad h' = \frac{dh}{dt}$$

and

$$\sqrt{h(t)} - \sqrt{h(t_0)} = -\frac{a}{A}\sqrt{\frac{g}{2}}(t - t_0)$$

which gives

$$h(t) = h_0(1 - \frac{t}{T})^2, \quad t_0 \leq t \leq t_0 + T$$

where $h(t_0) = h_0$, and $T = \dfrac{A}{a}\sqrt{2h_0}g$.

b. The equation

$$h(t) = h_0(1 - \frac{t}{T})^2, \quad 0 \le t \le T$$

can be rewritten as

$$y(t) = \frac{1}{1 - \sqrt{\dfrac{h(t)}{h_0}}} = \frac{1}{t}T = \varphi(t)\theta$$

c. The data are $h_0 = 10$ and

$$\begin{array}{lcccc} t = & 1 & 2 & 3 & 4 \\ h(t) = & 8.9 & 7.4 & 6.3 & 5.5 \\ y(t) = & 17.67 & 7.16 & 4.85 & 3.87 \\ \phi(t) = & 1 & 0.5 & 0.333 & 0.25 \end{array} \tag{7.32}$$

which yields the least-squares estimate

$$\hat{T} = \left(\sum_1^4 \varphi_k^2\right)^{-1} \sum_1^4 \varphi_k y_k = 16.7$$

7.5 The robot dynamics equations for τ_1 and τ_2 are linear in the mass parameters m_1 and m_2. Collecting terms gives

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \varphi^T \theta \tag{7.33}$$

with

$$\begin{aligned} \varphi_{11} &= l_1^2 \ddot{q}_1 + l_1 g c_1 && (7.34) \\ \varphi_{12} &= l_2^2(\ddot{q}_1 + \ddot{q}_2) + l_1 l_2 c_2 (2\ddot{q}_1 + \ddot{q}_2) + l_1^2 \ddot{q}_1 - l_1 l_2 s_2 \dot{q}_2^2 - 2 l_1 l_2 s_2 \dot{q}_1 \dot{q}_2 \\ &+ l_2 g c_{12} + l_1 g c_1 && (7.35) \\ \varphi_{21} &= 0 && (7.36) \\ \varphi_{22} &= l_1 l_2 c_2 \ddot{q}_1 + l_1 l_2 s_2 \dot{q}_1^2 + l_2 g c_{12} + l_2^2(\ddot{q}_1 + \ddot{q}_2) && (7.37) \end{aligned}$$

b. The accelerations $\ddot{q}_i$ occurs in the robot dynamics equations as $\alpha\ddot{q}_i$ and $c_i\ddot{q}_j = \cos(q_i)\ddot{q}_j$. Filtering of $\alpha\ddot{q}_i$ with a first order low pass filter (see Chapter 12 in *System Modeling and Identification* for details of this theory.)

$$\lambda = \frac{1}{1 + pT}$$

where $p = d/dt$, gives

$$\lambda\{\alpha\ddot{q}_j\} = \frac{1}{1 + pT}\alpha\ddot{q}_j = \alpha\frac{p}{1 + pT}\dot{q}_j$$

which is a realizable filtering of the speed $\dot{q}_j$. The terms $\cos(q_i)\ddot{q}_j$ cannot be handled in the same way. Instead we use the relationship

$$\cos(q_i)\ddot{q}_j = \frac{d}{dt}\left(\cos(q_i)\dot{q}_j\right) + \sin(q_i)\dot{q}_i\dot{q}_j$$

A low pass filtering now gives

$$\frac{1}{1+pT}\left(\cos(q_i)\ddot{q}_j\right) = \frac{p}{1+pT}\left(\cos(q_i)\dot{q}_j\right) + \frac{1}{1+pT}\left(\sin(q_i)\dot{q}_i\dot{q}_j\right)$$

where the right side is realizable. Introduce the filters

$$F_1 = \frac{1}{1+pT}, \tag{7.38}$$

$$F_2 = \frac{p}{1+pT} = \frac{1}{T}(1-\lambda) \tag{7.39}$$

Applying the filter F_1 on the robot dynamics equations, the result can be written as

$$\tau_f = \begin{pmatrix} \tau_{f_1} \\ \tau_{f_2} \end{pmatrix} = \begin{pmatrix} \varphi_{f_{11}} & \varphi_{f_{12}} \\ \varphi_{f_{21}} & \varphi_{f_{22}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \varphi_f^T \theta\,(\,) \tag{7.40}$$

where $\tau_f = F_1\tau$ and

$$\begin{aligned}
\varphi_{f_{11}} &= l_1^2 F_2\dot{q}_1 + l_1 g F_1 c_1 && (7.41)\\
\varphi_{f_{12}} &= l_2^2(F_2\dot{q}_1 + F_2\dot{q}_2) + l_1 l_2 2(F_2(c_2\dot{q}_1) + F_1(s_2\dot{q}_1\dot{q}_2)) && (7.42)\\
&+ l_1 l_2(F_2(c_2\dot{q}_2) + F_1(s_2\dot{q}_2^2)) + l_1^2 F_2\dot{q}_1 - l_1 l_2 F_1(s_2\dot{q}_2^2) && (7.43)\\
&- 2 l_1 l_2 F_1(s_2\dot{q}_1\dot{q}_2) + l_2 g F_1 c_{12} + l_1 g F_1 c_1 && (7.44)\\
\varphi_{f_{21}} &= 0 && (7.45)\\
\varphi_{f_{22}} &= l_1 l_2(F_2(c_2\dot{q}_1) + F_1(s_2\dot{q}_1\dot{q}_2)) + l_1 l_2 F_1(s_2\dot{q}_1\dot{q}_2) && (7.46)\\
&+ l_2 g F_1 c_{12} + l_2^2(F_2\dot{q}_1 + F_2\dot{q}_2 && (7.47)
\end{aligned}$$

c. The robot dynamics equations can be written as

$$\tau = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \varphi_{14} & \varphi_{15} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \varphi_{24} & \varphi_{25} \end{pmatrix} \begin{pmatrix} m_2 l_2^2 \\ m_2 l_1 l_2 \\ (m_1+m_2) l_1^2 \\ m_2 l_2 \\ (m_1+m_2) l_1 \end{pmatrix} = \varphi^T \theta \tag{7.48}$$

where φ_{ii} are functions of q_i, $\dot{q}_i$ and $\ddot{q}_i$. We see that the parameter vector now contains combinations of the original parameters m_1, m_2, l_1 and l_2.

Identification is thus possible if the nonlinear equations relating the identified parameters θ, and the original parameters, are solvable. One such alternative is to solve for

$$\begin{pmatrix} m_1 \\ m_2 \\ l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\theta_5^2}{\theta_3} - \dfrac{\theta_4^2}{\theta_1} \\ \dfrac{\theta_4^2}{\theta_1} \\ \dfrac{\theta_2}{\theta_4} \\ \dfrac{\theta_1}{\theta_4} \end{pmatrix} \tag{7.49}$$

7.6

a. Introducing the variables

$$a(t) = -\alpha(t)u(t) \tag{7.50}$$
$$v(t) = \alpha(t)u(t)c(t) - R(t) \tag{7.51}$$

the dynamic equation for the dissolved oxygen dynamics can be written as

$$\dot{y}(t) - a(t)y(t) = v(t) \tag{7.52}$$

Given $y(t_0)$, the solution is

$$y(t) = e^{\int_{t_0}^{t} a(\tau)d\tau} y(t_0) + \int_{t_0}^{t} e^{\int_{s}^{t} a(\tau)d\tau} v(s)ds$$

Sampling gives

$$y(kh+h) = e^{\int_{kh}^{kh+h} a(\tau)d\tau} y(kh) + \int_{kh}^{kh+h} e^{\int_{s}^{kh+h} a(\tau)d\tau} v(s)ds$$

Since $a(t)$ and $v(t)$ are constant between the sampling instants, this simplifies to

$$y(kh+h) = e^{a(kh)h} y(kh) + \int_0^h e^{a(kh)(h-s)} ds\, v(kh) \tag{7.53}$$

Introduce h^o by

$$h^o = \int_0^h e^{a(kh)(h-s)} ds = \frac{1}{a(kh)}(e^{a(kh)h} - 1)$$

This gives

$$e^{a(kh)h} = a(kh)h^o + 1$$

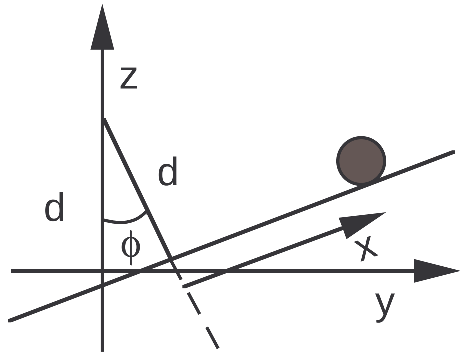

Figure 7.3 Definition of coordinates

Equation (7.53) can now be written as

$$\frac{y(kh+h)-y(kh)}{h^o} = a(kh)y(kh)+v(kh)$$

But (7.52) gives $a(kh)y(kh) + v(kh) = \dot{y}(kh)$ which gives the resulting sampled model

$$\dot{y}(kh) = \alpha(kh)u(kh)(c(kh)-y(kh)) - R(kh)$$

where

$$\dot{y}(kh) = \frac{y(kh+h)-y(kh)}{h^o} = \frac{y(kh+h)-y(kh)}{\frac{1}{a(kh)}(e^{a(kh)h}-1)} \tag{7.54}$$

b. We see from (7.54) that if h^o is replaced by the sampling interval h, we obtain a forward Euler approximation of the derivative. The sampling interval must typically be chosen smaller for the Euler approximation. This may lead to a sampling interval that is too small for identification of the relevant dynamics. If the assumption on piecewise constant signals is correct, the expression (7.54) gives an exact formula for $\dot{y}$, which gives more freedom in the choice of sampling interval.

7.7 Introducing the horizontal position y and the vertical position z for the ball (see Figure 7.3), we get

$$y = d\sin(\varphi) + x\cos(\varphi) \tag{7.55}$$
$$z = d - d\cos(\varphi) + x\sin(\varphi) \tag{7.56}$$

Differentiation gives

$$\ddot{y} = d\ddot{\varphi}\cos(\varphi) - d\dot{\varphi}^2\sin(\varphi) + \ddot{x}\cos(\varphi) \tag{7.57}$$
$$- \; 2\dot{x}\dot{\varphi}\sin(\varphi) - x\ddot{\varphi}\sin(\varphi) - x\dot{\varphi}^2\cos(\varphi) \tag{7.58}$$
$$\ddot{z} = d\ddot{\varphi}\sin(\varphi) + d\dot{\varphi}^2\cos(\varphi) + \ddot{x}\sin(\varphi) \tag{7.59}$$
$$+ \; 2\dot{x}\dot{\varphi}\cos(\varphi) + x\ddot{\varphi}\cos(\varphi) - x\dot{\varphi}^2\sin(\varphi) \tag{7.60}$$

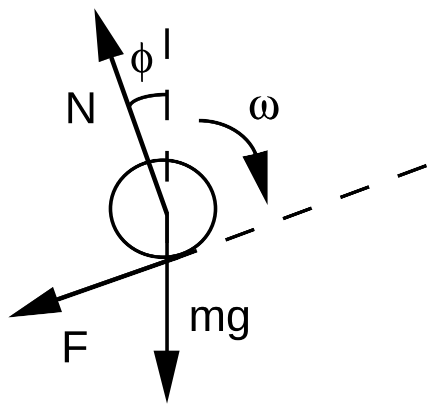

Figure 7.4 Forces acting on the ball

A force balance for the ball (see Fig. 7.4) is given by

$$m\ddot{y} = -N\sin(\varphi) - F\cos(\varphi) \tag{7.61}$$
$$m\ddot{z} = N\cos(\varphi) - F\sin(\varphi) - mg \tag{7.62}$$

The force N is eliminated by multiplying the upper equations in (7.60) and (7.61) with $\cos(\varphi)$, and multiplying the lower equations with $\sin(\varphi)$, resulting in

$$m(\ddot{y}\cos(\varphi) + \ddot{z}\sin(\varphi)) = -F - mg\sin(\varphi) = m(d\ddot{\varphi} + \ddot{x} - x\dot{\varphi}^2) \tag{7.63}$$

Assuming that the friction between the ball and the beam is such that the ball is rolling along the beam, the force F is given by

$$Fr = J_1\dot{\omega} = J_1\frac{\ddot{x}}{r} = \alpha m r^2\frac{\ddot{x}}{r} \;\Rightarrow\; F = \alpha m\ddot{x} \tag{7.64}$$

Using (7.63) and (7.64), we now obtain the dynamics from the angle φ of the beam, to the position x of the ball, as

$$(1+\alpha)\ddot{x} - x\dot{\varphi}^2 + d\ddot{\varphi} = -g\sin(\varphi) \tag{7.65}$$

An equation where also the applied torque τ is included is obtained from a torque balance for the beam:

$$J_2\ddot{\varphi} = \tau - Mgd\sin(\varphi) - Nx + Fd \tag{7.66}$$

Using (7.60) and (7.61), the force N is given by

$$N - mg\cos(\varphi) = m(\ddot{z}\cos(\varphi) - \ddot{y}\sin(\varphi)) \tag{7.67}$$
$$= m(d\dot{\varphi}^2 + 2\dot{x}\dot{\varphi} + x\ddot{\varphi}) \tag{7.68}$$

where the upper equations in (7.60) and (7.61) are multiplied by $-\sin(\varphi)$, and the lower equations are multiplied by $\cos(\varphi)$. The dynamic equation for the beam then becomes

$$J_2\ddot{\varphi} = \tau - Mgd\sin(\varphi) - m(d\dot{\varphi}^2 + 2\dot{x}\dot{\varphi} + x\ddot{\varphi} + g\cos(\varphi))x + \alpha m\ddot{x}d$$

Using (7.65) for replacing $x\dot{\varphi}^2$, we get

$$\begin{aligned} J_2\ddot{\varphi} &= \tau - Mgd\sin(\varphi) - m(d((1+\alpha)\ddot{x} + d\ddot{\varphi} + g\sin(\varphi)) + 2x\dot{x}\dot{\varphi} && (7.69)\\ &+ x^2\ddot{\varphi} + gx\cos(\varphi)) + \alpha m\ddot{x}d && (7.70) \end{aligned}$$

which is simplified to

$$\begin{aligned} J_2\ddot{\varphi} &= \tau - Mgd\sin(\varphi) && (7.71)\\ &- m\left(d\ddot{x} + d^2\ddot{\varphi} + dg\sin(\varphi) + 2x\dot{x}\dot{\varphi} + x^2\ddot{\varphi} + gx\cos(\varphi)\right) && (7.72) \end{aligned}$$

b. The measurable signals are x, $\dot{x}$, φ and τ. The dynamic equation (7.72) contains also $\dot{\varphi}$, $\ddot{\varphi}$, and $\ddot{x}$. The relation

$$2x\dot{x}\dot{\varphi} + x^2\ddot{\varphi} = \frac{d^2}{dt^2}(x^2\varphi) - 2\frac{d}{dt}(x\dot{x}\varphi)$$

can be used to rewrite the dynamics (7.72) as

$$\begin{aligned} J_2\ddot{\varphi} = \; & \tau - Mgd\sin(\varphi) - m(d\ddot{x} + d^2\ddot{\varphi} + dg\sin(\varphi) + gx\cos(\varphi) && (7.73)\\ & + \frac{d^2}{dt^2}(x^2\varphi) - 2\frac{d}{dt}(x\dot{x}\varphi)) && (7.74) \end{aligned}$$

and we see that by filtering both sides in (7.73) with

$$G_f(p) = \lambda^2 = \frac{1}{(1+pT)^2}$$

where $p = \dfrac{d}{dt}$, an identification model on the form

$$\tau_f = \varphi_f^T\theta$$

is obtained.

7.8 We consider a compartmental model which evolves according to the differential equation

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

with a matrix A containing the transfer coefficients which describe transfer from one compartment to another. Let us interpret the transfer of material between the compartments as the *flow* variable J, *i.e.*,

$$J(x(t)) = \dot{x}(t) = Ax(t), \quad x \in \mathbb{R}^n$$

Let us also introduce the candidate *potential* function as

$$V(x) = \frac{1}{2}x^T Px, \quad P = P^T > 0 \tag{7.75}$$

If $V(x)$ is the potential, then one would obtain the *force* variable as the gradient, *i.e.*,

$$F(x) = -\frac{\partial V(x)}{\partial x} = -Px$$

The equilibrium $x = 0$ is thus the point where the force $F(x)$ and the flow $J(x)$ both are zero. We also notice that

$$F^T(x)J(x) = -\frac{1}{2}x^T(PA + A^T P)x$$

It is well known in stability theory that for any stable system matrix A, *i.e.*, with all eigenvalues of A having negative real part, we are able to find a positive definite solution

$$P = P^T = \lim_{T\to\infty} \int_0^T e^{A^T t} Q e^{At} dt > 0$$

to the *Lyapunov equation*

$$PA + A^T P = -Q, \quad Q = Q^T > 0$$

Hence, for any stable system matrix A, *i.e.*, with all eigenvalues of A having negative real part, we are able to find a P of Eq. (7.75) which is a potential function. For such systems we notice that

$$F^T(x)J(x) = -\frac{1}{2}x^T(PA + A^T P)x = \frac{1}{2}x^T Qx \geq 0$$

In addition, the value of the potential $V(x(t))$ for stable systems decreases in the course of time as

$$\begin{aligned} V(x(t)) - V(x(t_0)) &= \int_{t_0}^{t} (\frac{\partial V}{\partial x})^T \dot{x}(t) dt && (7.76) \\ &= -\int_{t_0}^{t} F^T(x(t))J(x(t)) dt && (7.77) \\ &= -\int_{t_0}^{t} x^T(t)Qx(t) dt \leq 0 && (7.78) \end{aligned}$$

8

The Experimental Procedure

8.1 Consider a discrete-time sinusoidal sequence $\{u_k\}_{k=1}^{\infty}$ where the components $u_k = \sin \omega_0 k$. It has a periodic Fourier transform

$$U_\Delta(i\omega) = U(i\omega) \star \mathcal{F}\{\text{Ш}_h(t)\} = \frac{\pi}{i}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)) \star \frac{h}{2\pi}\text{Ш}_{2\pi/h}(\omega)$$

which results in

$$U_\Delta(i\omega) = \sum_{k=-\infty}^{\infty} \frac{\pi}{i}\left(\delta(\omega - \omega_0 + k\omega_s) - \delta(\omega + \omega_0 + k\omega_s)\right)$$

with $\omega_s = N\omega_0$.

The signal $u(k)$ is fed to the DA-converter which acts like a filter with impulse response

$$w(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases} \tag{8.1}$$

where $T = 2\pi/\omega_s$, and transfer function

$$W(i\omega) = \int_0^T e^{-i\omega t}\, dt = e^{-i\omega T/2}\frac{\sin \omega T/2}{\omega/2}.$$

The power of the fundamental frequency component is proportional to

$$P_0 = \frac{\sin^2 \omega_0/2T}{(\omega_0/2)^2} = \frac{\sin^2 \pi/N}{(\omega_0/2)^2}.$$

The strongest harmonic frequency component is the one at $\omega = \omega_s - \omega_0$. Its power is proportional to

$$P_1 = \frac{\sin^2(\omega_s - \omega_0)T/2}{(\omega_s - \omega_0)^2/4} = \frac{\sin^2 \pi(1-1/N)}{(\omega_0/2)^2(N-1)^2} = \frac{\sin^2 \pi/N}{(\omega_0/2)^2(N-1)^2}.$$

The total power in all the harmonic frequency components is proportional to

$$\begin{aligned}
P_h &= \sum_{k=1}^{\infty} \left(\frac{\sin^2(k\omega_s - \omega_0)T/2}{(k\omega_s - \omega_0)^2/4} + \frac{\sin^2(k\omega_s + \omega_0)T/2}{(k\omega_s + \omega_0)^2/4} \right) && (8.2)\\
&= \sum_{k=1}^{\infty} \left(\frac{\sin^2 \pi(k-1/N)}{(\omega_0/2)^2(kN-1)^2} + \frac{\sin^2 \pi(k+1/N)}{(\omega_0/2)^2(kN+1)^2} \right) && (8.3)\\
&= \sum_{k=1}^{\infty} \left(\frac{\sin^2 \pi/N}{(\omega_0/2)^2(kN-1)^2} + \frac{\sin^2 \pi/N}{(\omega_0/2)^2(kN+1)^2} \right) && (8.4)\\
&= \frac{\sin^2 \pi/N}{(\omega_0/2)^2} \frac{1}{N^2} \sum_{k=1}^{\infty} \left(\frac{1}{(k-1/N)^2} + \frac{1}{(k+1/N)^2} \right) = [\ 1/N \text{ small }]\\
&\approx \frac{\sin^2 \pi/N}{(\omega_0/2)^2} \frac{2}{N^2} \sum_{k=1}^{\infty} 1k^2 = \frac{\sin^2 \pi/N}{(\omega_0/2)^2} \frac{\pi^2}{3N^2} && (8.5)
\end{aligned}$$

a. If we require that the power in the strongest harmonic to be less than 1% of the power at the fundamental frequency

$$\frac{P_1}{P_0} = \frac{1}{(N-1)^2} < 0.01 \quad \Rightarrow \quad N > 11$$

b. If we require that more than 99% of the power appear at the fundamental frequency

$$\frac{P_h}{P_0 + P_h} = \frac{\pi^2}{3N^2 + \pi^2} < 0.01 \quad \Rightarrow \quad N > 18$$

We conclude that we need to sample the signal with a sampling frequency 20 times higher than that of the test frequency sinusoid.

8.2 Each value in the sequence $\{u_k\}$ is assumed to be normally distributed $\mathcal{N}(0, \sigma^2)$. Let $F(x)$ denote the normal distribution function

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

A confidence interval with an upper and a lower limit—*i.e.*, a two-sided test—is relevant(see Table B.1). The probability that $|u_k| > u_{lim}$ is given by

$$P < 2(1 - F(\frac{u_{lim}}{\sigma}))$$

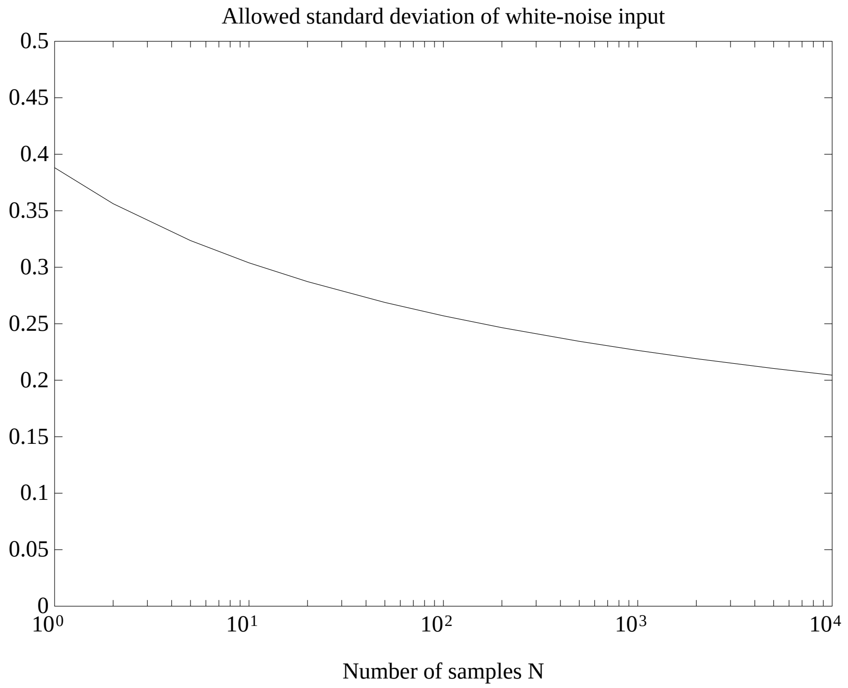

Figure 8.1 Allowed standard deviation σ of white-noise input $\{u_k\}$ where $u_k \in \mathcal{N}(0, \sigma^2)$ for various number of samples N.

To have $P < 0.01$, we have to have

$$F(\frac{u_{lim}}{\sigma}) > 0.995 \quad \Rightarrow \quad \sigma < u_{lim} 2.5758 = \frac{1}{2.5758} = 0.3882$$

This is the relevant answer if we want to assure that a particular $|u_k|$ does not exceed the limit the sequence $u_{lim} = 1$. However, if we want to assure that $\{|u_k|\}_{k=1}^N$ does not exceed the limit u_{lim} at any sample k with the probability P=0.01, then we should choose σ so that

$$[1 - 2(1 - F(\frac{u_{lim}}{\sigma}))]^N < P = 0.01$$

so that

$$F(\frac{u_{lim}}{\sigma}) = \frac{1 + P^{1/N}}{2}$$

or

$$\sigma = \frac{u_{lim}}{F^{-1}(\frac{1}{2}(1 + P^{1/N}))}$$

The upper limit of σ for a range of values of N is shown in Fig. 8.1.

8.3 Direct calculations give

$$m_2 = 1/3, \qquad m_3 = 1/7$$

and

$$C_2(\tau) = \begin{cases} 8/9 = 1 - 1/9, & \tau = 3k, \quad k = 0, \pm 1, \pm 2, \ldots \\ -4/9 = -1/3 - 1/9, & \text{otherwise} \end{cases} \tag{8.6}$$

$$C_3(\tau) = \begin{cases} 48/49 = 1 - 1/49, & \tau = 7k, \quad k = 0, \pm 1, \pm 2, \ldots \\ -8/49 = -1/7 - 1/49, & \text{otherwise} \end{cases} \tag{8.7}$$

and hence in general

$$C_N(\tau) = \begin{cases} 1 - 1/M^2, & \tau = Mk, \quad k = 0, \pm 1, \pm 2, \ldots \\ -1/M - 1/M^2, & \text{otherwise} \end{cases} \tag{8.8}$$

with $M = 2^N - 1$.

Even for moderately large values of N, M will be large and the term $1/M^2$ in $C_N(\tau)$ can be discarded. Then

$$C_N(\tau) = -\frac{1}{M} + \frac{M+1}{M} \sum_{k=-\infty}^{\infty} \partial(\tau + (2^N - 1)k)$$

hence

$$\begin{aligned} C_N(i\omega) &= -\frac{1}{M}\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\omega + k\omega_s) + \frac{M+1}{M}\frac{1}{MT} \sum_{k=-\infty}^{\infty} \delta(\omega + k\frac{\omega_s}{M}) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} c_k \delta(\omega + k\frac{\omega_s}{M}) \end{aligned} \tag{8.9}$$

with

$$c_k = \begin{cases} 1/M, & k = nM, \quad n = 0, \pm 1, \pm 2, \ldots \\ (M+1)/M^2, & \text{otherwise} \end{cases} \tag{8.10}$$

That is, the spectrum of a PRBS consists of a sum of sinusoids of (almost) equal amplitude spaced $\omega_s/(2^N - 1)$ apart.

8.4 A DA-converter can be thought of as a linear system with the impulse response

$$w(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \tag{8.11}$$

fed with a train of Dirac pulses where each pulse have an energy corresponding to the digital value fed to the converter (T is the sample period). The transfer function of the converter is

$$W(i\omega) = \int_0^T e^{-i\omega t}\, dt = e^{-i\omega T/2} \frac{\sin \omega T/2}{\omega/2}.$$

Therefore when feeding the PRBS through the DA-converter its spectrum is changed with

$$|W(i\omega)|^2 = \frac{\sin^2 \omega T/2}{(\omega/2)^2}$$

and although the PRBS has flat spectrum, the signal at the output of the DA-converter no longer has any flat spectrum.

8.5 The coherence function can be written

$$\gamma_{yu} = \sqrt{\frac{S_{yu}^2}{S_u S_y}} = \sqrt{\frac{|G|^2 S_u^2}{S_u(|G|^2 S_u + S_n)}} = \frac{1}{\sqrt{1 + \dfrac{S_n}{|G|^2 S_u}}}$$

Hence, γ_{yu} will be close to 1 at frequences where the effect of the input signal dominates over the disturbance, while it will approach 0 for frequences where the disturbance dominates.

8.6 The conclusion about low pass character is correct, but claiming a resonance at frequency 20 is dubious since the coherence function is almost 0 at that frequency.

8.7 A sinusoid is only exciting of order two and therefore not sufficient if three unknown parameters are to be consistently identified according to criteria of persistency of excitation.

8.8 The system that generates data is assumed to be

$$\begin{aligned} \mathcal{S}: \quad y_{k+1} &= -a y_k + b u_k + w_{k+1} + c w_k && (8.12) \\ u_k &= -K y_k && (8.13) \end{aligned}$$

which gives the closed-loop system

$$\mathcal{S}': \quad y_{k+1} = -(a + bK) y_k + w_{k+1} + c w_k$$

A least-squares estimate of $\alpha = -(a + bK)$ gives

$$\begin{aligned} \widehat{\alpha} &= (\Phi_N^T \Phi_N)^{-1} \Phi_N^T \mathcal{Y}_N && (8.14) \\ &= (\sum_{k=0}^{N-1} y_k^2)^{-1} (\sum_{k=0}^{N-1} -y_k y_{k+1}) && (8.15) \\ &= (\frac{1}{N-1} \sum_{k=0}^{N-1} y_k^2)^{-1} (\frac{1}{N-1} \sum_{k=0}^{N-1} -y_k y_{k+1}) && (8.16) \end{aligned}$$

The asymptotic results are

$$\begin{aligned}
\mathcal{E}\{y_{k+1}^2\} &= \mathcal{E}\{(-(a+bK)y_k + w_{k+1} + cw_k)^2\} && (8.17)\\
&= (a+bK)^2\mathcal{E}\{y_k^2\} + \sigma^2 + c^2\sigma^2 - 2(a+bK)c\sigma^2\} && (8.18)\\
\mathcal{E}\{y_{k+1}y_k\} &= \mathcal{E}\{(-(a+bK)y_k + w_{k+1} + cw_k)y_k\} && (8.19)\\
&= -(a+bK)\mathcal{E}\{y_k^2\} + c\sigma^2 && (8.20)\\
\mathcal{E}\{y_{k+2}y_k\} &= \mathcal{E}\{(-(a+bK)y_{k+1} + w_{k+2} + cw_{k+1})y_k\} && (8.21)\\
&= -(a+bK)\mathcal{E}\{y_{k+1}y_k\} && (8.22)
\end{aligned}$$

Under stationary conditions we have $\mathcal{E}\{y_{k+1}^2\} = \mathcal{E}\{y_k^2\}$ so that

$$\mathcal{E}\{y_k^2\} = \frac{\sigma^2}{1-(a+bK)^2}(1 + c^2 - 2c(a+bK))$$

and the asymptotical least-squares estimate is

$$\begin{aligned}
\mathcal{E}\{\widehat{\alpha}\} &= \mathcal{E}\{(\Phi_N^T\Phi_N)^{-1}\Phi_N^T\mathcal{Y}_N\} && (8.23)\\
&= \mathcal{E}\{(\sum_{k=0}^{N-1} y_k^2)^{-1}(\sum_{k=0}^{N-1} -y_ky_{k+1})\} && (8.24)\\
&= \mathcal{E}\{(\frac{1}{N-1}\sum_{k=0}^{N-1} y_k^2)^{-1}(\frac{1}{N-1}\sum_{k=0}^{N-1} -y_ky_{k+1})\} && (8.25)\\
&= \frac{1}{\mathcal{E}\{y_k^2\}}(-(a+bK)\mathcal{E}\{y_k^2\} + c\sigma^2) && (8.26)\\
&= (a+bK) - \frac{c\sigma^2}{\mathcal{E}\{y_k^2\}} && (8.27)\\
&= (a+bK) - \frac{c\sigma^2}{\dfrac{\sigma^2}{1-(a+bK)^2}(1+c^2-2c(a+bK))} && (8.28)
\end{aligned}$$

which is biased as expected.

By solving the Yule-Walker equations

$$\begin{pmatrix} -\mathcal{E}\{y_{k+1}y_k\} & 0 \\ -\mathcal{E}\{y_k^2\} & 1 \end{pmatrix}\begin{pmatrix} a+bK \\ c\sigma^2 \end{pmatrix} = \begin{pmatrix} \mathcal{E}\{y_{k+2}y_k\} \\ \mathcal{E}\{y_{k+1}y_k\} \end{pmatrix} \tag{8.29}$$

or

$$\begin{pmatrix} -C_{yy}(1) & 0 \\ -C_{yy}(0) & 1 \end{pmatrix}\begin{pmatrix} a+bK \\ c\sigma^2 \end{pmatrix} = \begin{pmatrix} C_{yy}(2) \\ C_{yy}(0) \end{pmatrix} \tag{8.30}$$

with the solution

$$\begin{pmatrix} a+bK \\ c\sigma^2 \end{pmatrix} = \begin{pmatrix} -\mathcal{E}\{y_{k+1}y_k\} & 0 \\ -\mathcal{E}\{y_k^2\} & 1 \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{E}\{y_{k+2}y_k\} \\ \mathcal{E}\{y_{k+1}y_k\} \end{pmatrix} \tag{8.31}$$

or—using estimates $\widehat{C}_{yy}(\tau)$—we have

$$\begin{pmatrix} \widehat{a+bK} \\ \widehat{c\sigma^2} \end{pmatrix} = \begin{pmatrix} -\widehat{C}_{yy}(1) & 0 \\ -\widehat{C}_{yy}(0) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{C}_{yy}(2) \\ \widehat{C}_{yy}(0) \end{pmatrix} \tag{8.32}$$

$$= \frac{1}{C_{yy}(1)} \begin{pmatrix} -\widehat{C}_{yy}(2) \\ \widehat{C}_{yy}(1)^2 - \widehat{C}_{yy}(0)\widehat{C}_{yy}(2) \end{pmatrix} \tag{8.33}$$

8.9 We use the correlation matrix $R_{uu}(n)$ as formulated in the context of persistency of excitation (see *System Modeling and Identification* Sec. 8.5). For the step-formed input we have

$$R_{uu}(n) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}, \quad R_{uu}(n) \in \mathbb{R}^{n\times n} \tag{8.34}$$

with rank $R_{uu}(n) = 1$ for all orders n. Hence we have persistency of excitation of order one. Using such a step-formed input in an identification experiment we can expect one parameter only—for instance, a static gain—to be consistently estimated.

REMARK: To avoid possible confusion arising we have to remind the reader that statistical consistency deals with asymptotic properties and not with transients. Traditional step-response tests rely on the information received in the course of a transient behaviour of the system whereas the stationary properties of a step response only give information about the static gain of the system.

8.10 We consider a continuous-flow fermentation process can be modeled by the equations

$$\dot{x} = \mu x - i_{in} x \tag{8.35}$$

$$\dot{s} = -R\mu x + i_{in}(s_{in} - s) \tag{8.36}$$

where x is the produced biomass, s substrate concentration, s_{in} influent substrate concentration, i_{in} influent flow rate, R yield coefficient, μ specific growth rate.

The first problem in this context is that this is a nonlinear equation if we consider i_{in} to be a control input. On the other hand, if we assume i_{in} to be constant, then we have a linear but autonomous system, *i.e.*, a system without any control input. Second, according to Eq. (8.35) we find that x increases exponentially for $i_{in} < \mu$. Third, for such an unstable system we anticipate problems with the system's initial condition which somehow has to be estimated. One approach to solve this problem is to formulate linear regression models in the form of integrals over time intervals $[t,t_1]$,

$$x(t_{k+1}) - x(t_0) = \mu \int_{t_k}^{t_{k+1}} x(t)dt - \int_{t_k}^{t_{k+1}} i_{in} x(t)dt \tag{8.37}$$

$$s(t_{k+1}) - s(t_k) = -R\mu \int_{t_k}^{t_{k+1}} x(t)dt + \int_{t_k}^{t_{k+1}} i_{in}(s_{in} - s(t))dt \tag{8.38}$$

9

Model Validation

9.1 The augmented system equation for the error of the least-squares estimate is

$$\begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \widetilde{\theta}_N \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{9.1}$$

so that

$$\begin{pmatrix} \varepsilon \\ \widetilde{\theta}_N \end{pmatrix} = \begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{9.2}$$

A covariance calculation gives

$$\mathrm{Cov}\{\begin{pmatrix} e \\ 0 \end{pmatrix}\} = \begin{pmatrix} \Sigma_e & 0 \\ 0 & 0 \end{pmatrix} \tag{9.3}$$

Thus, we can estimate

$$\mathrm{Cov}\{\begin{pmatrix} \varepsilon \\ \widetilde{\theta}_N \end{pmatrix}\} = \begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix}^{-1} \tag{9.4}$$

9.2 A first calculation is

$$\begin{pmatrix} I & \Phi_N \\ \Phi_N^T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T & \Phi_N(\Phi_N^T\Phi_N)^{-1} \\ (\Phi_N^T\Phi_N)^{-1}\Phi_N^T & -(\Phi_N^T\Phi_N)^{-1} \end{pmatrix} \tag{9.5}$$

when $\Phi_N^T\Phi_N$ is invertible. Direct application to the covariance expression in Exercise 9.1 gives

$$(I - \Phi_N(\Phi_N^T\Phi_N)^{-1}\Phi_N^T)\Sigma_e\Phi_N(\Phi_N^T\Phi_N)^{-1}$$

In the special case when $\Sigma_e = \sigma^2 I$ we note that the covariance between ε and $\widetilde{\theta}_N$ is zero.

10

Model Approximation

10.1 A polynomial series expansion of $G_2(s)$ is

$$G_2(s) = \frac{\beta}{\alpha} + \frac{1}{\alpha}(1 - \beta - \beta\alpha)s + \cdots$$

Matching of the truncated Taylor expansion to an mth order transfer function gives

$$B_m(s) = G_m(s)A_m(s)$$

For $m = 1$ we have

$$b = (g_0 + g_1 s)(s + a)$$

so that

$$\begin{cases} a = -g_0/g_1 = -\dfrac{\beta}{\alpha}/(\dfrac{1}{\alpha}(1 - \beta - \beta\alpha)) \\ b = g_0 a = -g_0^2/g_1 = -(\dfrac{\beta}{\alpha})^2/(\dfrac{1}{\alpha}(1 - \beta - \beta\alpha)) \end{cases} \tag{10.1}$$

10.2 As σ_1 and σ_2 are of the same order of magnitude it is not advisable to do any model reduction. Factorization of the transfer function

$$\frac{z - 1}{z^2 - 1.79z + 0.792} = \frac{z - 1}{(z - 0.99)(z - 0.8)}$$

shows that the zero at $z = 1$ "almost" cancels the pole at $z = 0.99$.

10.3 Let $z = \bar{x} = T^{-1}x$ and apply the theory presented in Sec. 10.2 of *System Modeling and Identification* [1]. We find the T^{-1} that diagonalizes

$$\bar{P} = T^{-1}PT^{-T} \tag{10.2}$$

$$\bar{Q} = T^T Q T \tag{10.3}$$

10.4 Assuming that the balanced state-space representation of a certain given continuous-time system is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{10} \\ A_{01} & A_{00} \end{pmatrix}\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_0 \end{pmatrix} u \tag{10.4}$$

$$y = \begin{pmatrix} C_1 & C_0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} + Du \tag{10.5}$$

one can approach model reduction by approximating

$$\dot{x}_0 = 0, \quad \text{and} \quad x_0 = -A_{00}^{-1}A_{01}x_1 - A_{00}^{-1}B_0u \tag{10.6}$$

The reduced-order state-space model is then

$$\dot{x}_1 = (A_{11} - A_{10}A_{00}^{-1}A_{01})x_1 + (B_1 - A_{10}A_{00}^{-1}B_0)u \tag{10.7}$$

$$y = (C_1 - C_0A_{00}^{-1}A_{01})x_1 + (D - C_0A_{00}^{-1}B_0)u \tag{10.8}$$

which contains a direct term from u to y also if the full-order direct term is zero.

Now assume that we define X according to

$$\begin{matrix} x_0 \in \mathbb{R}^m \\ x_1 \in \mathbb{R}^n \\ u \in \mathbb{R}^p \end{matrix} \qquad \text{and} \quad X = \begin{pmatrix} x_0 \\ x_1 \\ u \end{pmatrix} \in \mathbb{R}^{m+n+p} \tag{10.9}$$

From Eq. (10.6) we find that the states of the reduced-order model evolve on the subspace determined by the the equation

$$\begin{pmatrix} A_{00}^{-1}A_{01} & I_{m\times m} & A_{00}^{-1}B_0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_0 \\ u \end{pmatrix} = 0 \tag{10.10}$$

which we may denote

$$A_mX = 0, \qquad A_m \in \mathbb{R}^{m\times(m+n+p)}$$

A suitable projection matrix is

$$P = I_{(m+n+p)\times(m+n+p)} - A_m^T(A_mA_m^T)^{-1}A_m$$

The system output can thus be written

$$y = \begin{pmatrix} C_1 & C_0 & D \end{pmatrix}\begin{pmatrix} x_1 \\ x_0 \\ u \end{pmatrix} \tag{10.11}$$

and the output of the reduced-order model

$$y_r = \begin{pmatrix} C_1 & C_0 & D \end{pmatrix} PX = \begin{pmatrix} C_1 & C_0 & D \end{pmatrix} P \begin{pmatrix} x_1 \\ x_0 \\ u \end{pmatrix} \tag{10.12}$$

which gives the desired geometric interpretation of projection of the original $(m+n+p)$—dimensional state vector to a $(n+p)$—dimensional subspace.

10.5 We consider a state vector x with the system equation

$$\dot{x} = Ax + Bu$$

and the impulse response

$$x(t) = e^{At}B, \qquad t \geq 0$$

The following integral is essential in order to evaluate the energy of x

$$e_{xx} = \int_0^\infty x(t)x^T(t)dt = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$$

If we introduce

$$P(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$$

with the derivative

$$\frac{dP(t)}{dt} = e^{At}BB^Te^{A^Tt}dt$$

we also notice that

$$\begin{aligned}
P(t)A^T + AP(t) &= \int_0^t e^{A\tau}BB^Te^{A^T\tau}A^T + Ae^{A\tau}BB^Te^{A^T\tau}d\tau && (10.13)\\
&= \int_0^t \frac{d}{d\tau}e^{A\tau}BB^Te^{A^T\tau}d\tau && (10.14)\\
&= e^{At}BB^Te^{A^Tt} - BB^T && (10.15)\\
&= \frac{dP(t)}{dt} - BB^T && (10.16)
\end{aligned}$$

and we can summarize the matrix differential equation

$$\frac{dP(t)}{dt} = P(t)A^T + AP(t) + BB^T$$

For a stable system we expect to have the limit

$$\lim_{t\to\infty}\frac{dP(t)}{dt} = 0 \quad \Rightarrow \quad 0 = P(\infty)A^T + AP(\infty) + BB^T$$

which satisfies the Lyapunov equation

$$PA^T + AP = -BB^T$$

Assume now that we consider a state-space transformation

$$z = Tx, \qquad \text{with} \quad \dot{z} = TAT^{-1}x + TBu$$

with the impulse response energy

$$e_{zz} = \int_0^\infty z(t)z^T(t)dt = P_z$$

where P_z solves the Lyapunov equation

$$P_z T^{-T} A^T T^T + TAT^{-1} P_z = -TBB^T T^T$$

By multiplication from the left by T^{-1} and T^{-1} we have

$$T^{-1} P_z T^{-T} A^T + AT^{-1} P_z T^{-T} = -BB^T$$

so that

$$P = T^{-1} P_z T^{-T}$$

By choosing T as a matrix factor of P^{-1} obtained from the matrix factorization equation

$$P^{-1} = T^T T$$

we manage to find the state-space transformation

$$z = Tx$$

with the finite energy integral

$$e_{zz} = \int_0^\infty z(t)z^T(t)dt = P_z = I$$

and we may conclude that the impulse responses in the state vector z are "orthogonal."

EXAMPLE: Consider the state-space system

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \tag{10.17}$$

A solution to the Lyapunov equation

$$P \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}^T + \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} (\,)\, P = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{10.18}$$

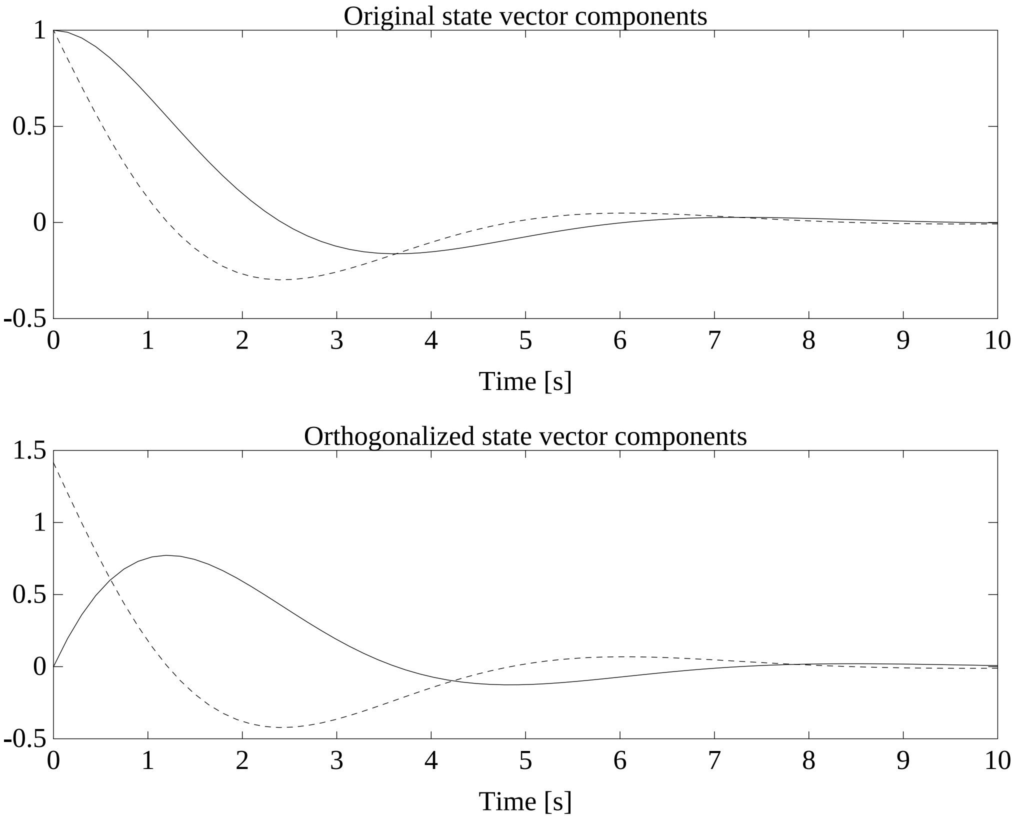

Figure 10.1 Ortogonalized impulse resonses by means of a state space transformation derived from the Lyapunov equation.

is

$$P = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \tag{10.19}$$

and

$$T = \sqrt{2} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{10.20}$$

solves the equation $P^{-1} = T^T T$. The state-space transformation

$$z = Tx, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{10.21}$$

has components z_1 and z_2 with orthogonal impulse responses, *i.e*,

$$\int_0^\infty z_i(t) z_j(t) dt = \delta_{ij}, \qquad i, j = 1, 2$$

Orthogonal signals can be applied in order to produce test signals with suitable excitation properties.

11

Real-Time Identification

11.1 The prediction error least-squares criterion applied to recursive least-squares identification is

$$V(\widehat{\theta}_k) = \frac{1}{2}(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k)^T(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k) = \frac{1}{2}\varepsilon(\widehat{\theta}_k)^T\varepsilon(\widehat{\theta}_k)$$

and a weighted parameter error criterion is

$$Q(\widehat{\theta}_k) = \frac{1}{2}(\widehat{\theta}_k - \theta)^T(\Phi_k^T\Phi_k)(\widehat{\theta}_k - \theta) = \frac{1}{2}\widetilde{\theta}_k^T(\Phi_k^T\Phi_k)\widetilde{\theta}_k = \frac{1}{2}\widetilde{\theta}_k^T P_k^{-1}\widetilde{\theta}_k$$

The prediction error least-squares criterion can be expressed as

$$\begin{aligned} V(\widehat{\theta}_k) &= \frac{1}{2}(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k)^T(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k) \\ &= \frac{1}{2}(-\Phi_k\widetilde{\theta}_k + v)^T(-\Phi_k\widetilde{\theta}_k + v) \\ &= \frac{1}{2}(\widetilde{\theta}_k^T\Phi_k^T\Phi_k\widetilde{\theta}_k + v^Tv - 2\widetilde{\theta}^T\Phi_k^Tv) \qquad (11.1) \end{aligned}$$

The orthogonality principle states that

$$0 = \varepsilon(\widehat{\theta}_k)^T\Phi_k = v^T\Phi_k - \widetilde{\theta}_k^T\Phi_k^T\Phi_k \qquad (11.2)$$

Substitution of Eq. (11.2) into Eq. (11.1) and rearrangement of the terms gives the requested relationship

$$V(\widehat{\theta}_k) + Q(\widehat{\theta}_k) = v^Tv$$

11.2 The recursive least-squares algorithm including a forgetting factor for estimation of time-varying parameters is

$$\begin{aligned} \widehat{\theta}_k &= \widehat{\theta}_{k-1} + P_k\phi_k\varepsilon_k \qquad (11.3) \\ \varepsilon_k &= y_k - \phi_k^T\widehat{\theta}_{k-1} \qquad (11.4) \\ P_k &= \frac{1}{\lambda}(P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^TP_{k-1}}{\lambda + \phi_k^TP_{k-1}\phi_k}) \qquad (11.5) \end{aligned}$$

as presented in Sec. 11.2 of the textbook. The prediction error least-squares criterion and a parameter error criterion applied to recursive least-squares identification with forgetting factor λ are

$$V(\widehat{\theta}_k) = \frac{1}{2}(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k)^T W_k(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k) = \frac{1}{2}\varepsilon(\widehat{\theta}_k)^T W_k \varepsilon(\widehat{\theta}_k) \quad (11.6)$$

$$Q(\widehat{\theta}_k) = \frac{1}{2}(\widehat{\theta}_k - \theta)^T W_k(\Phi_k^T\Phi_k)(\widehat{\theta}_k - \theta) = \frac{1}{2}\widetilde{\theta}_k^T P_k^{-1}\widetilde{\theta}_k \quad (11.7)$$

where W_k is a weighting matrix with components

$$w_{ij} = \lambda^{k-i}\delta_{ij}, \qquad i = 1,2,\ldots,k \quad \text{and} \quad j = 1,2,\ldots,k$$

The prediction error least-squares criterion can be expressed as

$$\begin{aligned} V(\widehat{\theta}_k) &= \frac{1}{2}(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k)^T W_k(\mathcal{Y}_k - \Phi_k\widehat{\theta}_k) && (11.8)\\ &= \frac{1}{2}(-\Phi_k\widetilde{\theta}_k + v)^T W_k(-\Phi_k\widetilde{\theta}_k + v) && (11.9)\\ &= \frac{1}{2}(\widetilde{\theta}_k^T\Phi_k^T W_k\Phi_k\widetilde{\theta}_k + v^T W_k v - 2\widetilde{\theta}^T\Phi_k^T W_k v) && (11.10) \end{aligned}$$

The orthogonality principle states that

$$0 = \varepsilon(\widehat{\theta}_k)^T W_k\Phi_k = v^T W_k\Phi_k - \widetilde{\theta}_k^T\Phi_k^T W_k\Phi_k \quad (11.11)$$

Substitution of Eq. (11.11) into Eq. (11.10) and rearrangement of the terms gives the requested relationship

$$V(\widehat{\theta}_k) + Q(\widehat{\theta}_k) = v^T W_k v$$

As can be seen from the following calculation, the parameter error develops irregularly over time

$$\begin{aligned} Q(\widehat{\theta}_k) - \lambda Q(\widehat{\theta}_{k-1}) &= \frac{\lambda}{2}\widetilde{\theta}_k^T P_k^{-1}\widetilde{\theta}_k - \frac{1}{2}\widetilde{\theta}_{k-1}^T P_{k-1}^{-1}\widetilde{\theta}_{k-1} = && (11.12)\\ &= \frac{1}{2}\widetilde{\theta}_{k-1}^T(P_k^{-1} - \lambda P_{k-1}^{-1})\widetilde{\theta}_{k-1} && (11.13)\\ &+ \widetilde{\theta}_{k-1}^T\phi_k\varepsilon_k + \frac{1}{2}\phi_k^T P_k\phi_k\varepsilon_k^2 = && (11.14)\\ &= \frac{1}{2}(\widetilde{\theta}_{k-1}^T\phi_k + \varepsilon_k)^2 + \frac{1}{2}(-1 + \phi_k^T P_k\phi_k)\varepsilon_k^2 && (11.15)\\ &= \frac{1}{2}(\widetilde{\theta}_{k-1}^T\phi_k + \varepsilon_k)^2 - \frac{1}{2}\frac{\lambda}{\lambda + \phi_k^T P_{k-1}\phi_k}\varepsilon_k^2 && (11.16) \end{aligned}$$

Under the linear model assumption $y_k = \phi_k^T\theta + v_k$ so that $\varepsilon_k = -\widetilde{\theta}_{k-1}^T\phi_k + v_k$ one can conclude that the parameter error develops in the following indefinite way

$$Q(\widehat{\theta}_k) - \lambda Q(\widehat{\theta}_{k-1}) = \frac{1}{2}v_k^2 - \frac{1}{2}\frac{\lambda}{\lambda + \phi_k^T P_{k-1}\phi_k}\varepsilon_k^2 \quad (11.17)$$

whereas

$$V(\widehat{\theta}_k) + Q(\widehat{\theta}_k) = v^T W_k v$$

and

$$\mathcal{E}\{V(\widehat{\theta}_k) + Q(\widehat{\theta}_k)\} = \mathcal{E}\{v^T W_k v\} = \mathrm{tr}(W_k \mathcal{E}\{vv^T\}) = \mathrm{tr}(W_k \Sigma_v)$$

In particular, if $\Sigma_v = \sigma^2 I$ and we recall that the weighting matrix W_k with components $w_{ij} = \lambda^{k-i}\delta_{ij}$, then

$$\mathcal{E}\{V(\widehat{\theta}_k) + Q(\widehat{\theta}_k)\} = \sigma^2 \sum_{k=1}^{N} \lambda^{N-i} = \sigma^2 \frac{1-\lambda^{N+1}}{1-\lambda} < \frac{\sigma^2}{1-\lambda} \tag{11.18}$$

Finally, if we impose the condition of persistent excitation of the form

$$R = R^T = \lim_{k\to\infty} \frac{1}{k}\Phi_k^T W_k \Phi_k = \lim_{k\to\infty} \frac{1}{k} P_k^{-1} > 0$$

we can conclude from Eq. (11.18) that

$$\mathcal{E}\{\frac{1}{N}V(\widehat{\theta}_k) + \frac{1}{N}Q(\widehat{\theta}_k)\} = \frac{1}{N}\frac{\sigma^2}{1-\lambda}$$

so that

$$0 \le \lim_{k\to\infty} \mathcal{E}\{\frac{1}{k}V(\widehat{\theta}_k)\} = \lim_{k\to\infty} \mathcal{E}\{\frac{1}{2k}\widetilde{\theta}^T \Phi_k^T W_k \Phi_k \widetilde{\theta}\} \le \lim_{k\to\infty} \frac{1}{k} Q(\widehat{\theta}_k) = 0$$

As the weighting matrix P_k/k converges to the nonzero matrix R under conditions of persistent excitation, we can in such cases claim that $\lim_{k\to\infty} \|\widetilde{\theta}\| = 0$, *i.e.*, we claim to have consistent estimates of θ. However, as the condition of persistent excitation is difficult to check, this information is not very helpful in application.

11.3 We consider the variables

$$\begin{aligned} V(\widehat{\theta}_k) &= \frac{1}{2}\varepsilon(\widehat{\theta}_k)^T \varepsilon(\widehat{\theta}_k) & (11.19)\\ Q(\widehat{\theta}_k) &= \frac{1}{2}(\widehat{\theta}_k - \theta)^T P_k^{-1}(\widehat{\theta}_k - \theta) = \frac{1}{2}\widetilde{\theta}_k^T P_k^{-1}\widetilde{\theta}_k & (11.20)\end{aligned}$$

which satisfy the relationship

$$V(\widehat{\theta}_k) + Q(\widehat{\theta}_k) = v^T v$$

The mathematical expectation

$$\mathcal{E}\{V(\widehat{\theta}_k) + Q(\widehat{\theta}_k)\} = \mathcal{E}\{v^T v\} = \mathcal{E}\{\mathrm{tr}(vv^T)\} = \mathrm{tr}(\Sigma_v)$$

where Σ_v is the covariance matrix of the noise v under the assumption of a zero-mean noise sequence. For instance, assuming the uncorrelated noise sequence $\{v_k\}_{k=1}^N$ has independent identically distributed components with $\Sigma_v = \sigma^2 I$, we find that

$$\mathcal{E}\{v^T v\} = \mathcal{E}\{\mathrm{tr}\{vv^T\}\} = \mathrm{tr}\{\Sigma_v\} = N\sigma^2$$

and

$$\mathcal{E}\{V(\widehat{\theta}_k)\} + \mathcal{E}\{Q(\widehat{\theta}_k)\} = \mathcal{E}\{V(\widehat{\theta}_k)\} + \mathcal{E}\{\mathrm{tr}(P_k^{-1}\widetilde{\theta}_k\widetilde{\theta}_k^T)\} \tag{11.21}$$

As $\mathcal{E}\{\widetilde{\theta}_k\widetilde{\theta}_k^T\} = P_k\sigma^2$ we find for the last term of Eq. (11.21) is

$$\mathcal{E}\{\mathrm{tr}(P_k^{-1}\widetilde{\theta}_k\widetilde{\theta}_k^T)\} = \sigma^2\mathrm{tr}(I_{p\times p})$$

Hence we have the equation

$$\mathcal{E}\{V(\widehat{\theta}_k)\} = (N-p)\sigma^2$$

Based on this relationship we suggest the variance estimate

$$\widehat{\sigma}^2 = \frac{1}{N-p}V(\widehat{\theta}_k)$$

11.4 Consider the weighted parameter error

$$V(\widehat{\theta}_k) = \frac{1}{2}(\widehat{\theta}_k - \theta)^T Q^{-1}(\widehat{\theta}_k - \theta)$$

in the case of an algorithm

$$\begin{aligned}
\widehat{\theta}_k &= \widehat{\theta}_{k-1} + \gamma_k\varepsilon_k && (11.22)\\
\varepsilon_k &= y_k - \phi_k^T\widehat{\theta}_{k-1} && (11.23)\\
\gamma_k &= Q\phi_k/r_k, \qquad Q = Q^T > 0 && (11.24)\\
r_k &= r_{k-1} + \phi_k^T Q^{-1}\phi_k && (11.25)
\end{aligned}$$

where Q is some positive definite weighting matrix.

Assuming the system generating data can be described by the linear regression model $y_k = \phi_k^T\theta + v_k$, we now consider how $V(\widehat{\theta}_k)$ develops over time

$$
\begin{aligned}
V(\widehat{\theta}_k) - V(\widehat{\theta}_{k-1}) &= \frac{1}{2}\widetilde{\theta}_k^T Q^{-1}\widetilde{\theta}_k - \frac{1}{2}\widetilde{\theta}_{k-1}^T Q^{-1}\widetilde{\theta}_{k-1} && (11.26)\\
&= \frac{1}{2}(\widetilde{\theta}_{k-1} + \gamma_k\varepsilon_k)^T Q^{-1}(\widetilde{\theta}_{k-1} + \gamma_k\varepsilon_k)^T && (11.27)\\
&- \frac{1}{2}\widetilde{\theta}_{k-1}^T Q^{-1}\widetilde{\theta}_{k-1} && (11.28)\\
&= \varepsilon_k\gamma_k^T Q^{-1}\widetilde{\theta}_{k-1} + \frac{1}{2}\varepsilon_k^2\gamma_k^T Q^{-1}\gamma_k && (11.29)\\
&= \varepsilon_k(-\varepsilon_k + v_k)\frac{1}{r_k} + \frac{1}{2}\phi_k^T Q\phi_k\frac{\varepsilon_k^2}{r_k^2} && (11.30)\\
&= \frac{\varepsilon_k^2}{r_k^2}(-r_{k-1} + \phi_k^T(Q - Q^{-1})\phi_k) + \frac{\varepsilon_k v_k}{r_k} && (11.31)\\
&= \frac{\varepsilon_k^2}{r_k^2}(\phi_k^T(Q - Q^{-1})\phi_k) && (11.32)\\
&- r_{k-1}(\frac{\varepsilon_k}{r_k} - \frac{1}{2}\frac{v_k}{r_{k-1}})^2 + \frac{v_k^2}{4r_{k-1}} && (11.33)
\end{aligned}
$$

In the disturbance-free case with $v_k = 0$ for $k = 1, 2, \ldots$ we may conclude that $V(\widehat{\theta}_k)$ converges to zero if $Q < Q^{-1}$. Moreover, modification of the textbook algorithm to

$$
\begin{aligned}
\widehat{\theta}_k &= \widehat{\theta}_{k-1} + \gamma_k\varepsilon_k && (11.34)\\
\varepsilon_k &= y_k - \phi_k^T\widehat{\theta}_{k-1} && (11.35)\\
\gamma_k &= Q\phi_k/r_k, \qquad Q = Q^T > 0 && (11.36)\\
r_k &= r_{k-1} + \phi_k^T Q\phi_k && (11.37)
\end{aligned}
$$

is suitable in order to eliminate the restriction $Q < Q^{-1}$

12

Continuous-Time Models

12.1 The DC-motor transfer function is

$$G_0(s) = \frac{K}{Js + D} = \frac{K/J}{s + D/J}$$

expressed in the the gain K, the moment of inertia J and the damping D.

a. Introduce the operator translation

$$\lambda = \frac{1}{1 + s\tau}$$

so that

$$s = \frac{1}{\tau}\frac{1-\lambda}{\lambda} \tag{12.1}$$

Substitution of Eq. (12.1) into the transfer function gives

$$G_0(\lambda) = \frac{K}{J\frac{1}{\tau}\frac{1-\lambda}{\lambda} + D} = \frac{(K\tau/J)\lambda}{1 + (-1 + \frac{\tau D}{J})\lambda}$$

and a suitable linear regression model is

$$y(t) = (-1 + \frac{\tau D}{J})\lambda\{y\} + \frac{K\tau}{J}\lambda\{u\} \tag{12.2}$$

By determining the coefficents of Eq. (12.2) it is clear that it is possible to to identify the parameters K/J and D/J. However, as multiplication of the transfer function numerator and denominator by any nonzero number does not alter the original transfer function, it is clear already from Eq. (12.1) that we cannot hope to estimate all three system parameters. Thus, two of the three physical parameters are uniquely identifiable.

12.2

a. Assuming that there is a sinusoidal disturbance acting on the system

$$Y(s) = -\alpha\lambda(s)Y(s) + \beta\lambda(s)U(s) + \frac{1}{s^2+\omega^2}W(s)$$

where $W(s)$ represents white noise with constant spectral density. Now introduce the notch filter

$$F(s) = \frac{s^2+\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}$$

where ω_0 is chosen with respect to the intended sampling rate so that $\omega_0 < \omega_N$. Now introduce the filtered variables

$$\begin{aligned} Y_f(s) &= F(s)Y(s) \qquad (12.3)\\ U_f(s) &= F(s)U(s) \qquad (12.4)\end{aligned}$$

which satisfy the relationship

$$Y_f(s) = -\alpha\lambda(s)Y_f(s) + \beta\lambda(s)U_f(s) + \frac{1}{s^2+2\xi\omega_0 s+\omega_0^2}W(s)$$

Thus, application of the notch filter yields a linear regression model with a spectral density of the noise that is close to constant up to the Nyquist frequency ω_N. In addition, we may assume little bias of parameter estimates to appear in least-squares estimation of α and β.

b. Sampling of the variables

$$\begin{matrix} y(t)\\ \lambda\{u\}(t)\\ \lambda\{y\}(t)\end{matrix} \qquad (12.5)$$

for the sequence of time instants $t_1, t_2, t_3, \ldots$. Application of the following recursive identification algorithm

$$\begin{aligned} \varepsilon(t_k) &= y(t_k) - (-\lambda\{y\}(t_k) \quad \lambda\{u\}(t_k))\begin{pmatrix}\widehat{\alpha}\\ \widehat{\beta}\end{pmatrix} = \phi_k^T\widehat{\theta}_{k-1} \qquad (12.6)\\ P_k &= P_{k-1} - \frac{P_{k-1}\phi_k\phi^T P_{k-1}}{1+\phi_k^T P_{k-1}\phi_k} \qquad (12.7)\\ \widehat{\theta}_k &= \widehat{\theta}_{k-1} + P_k\phi_k\varepsilon(t_k) \qquad (12.8)\end{aligned}$$

```
i=sqrt(-1);
b=theta(2);
a=theta(1);
Nplot=300;
w=logspace(-2.5,0.5,Nplot)';
for k=1:Nplot,
  G(k,1)=b/(i*w(k) + a);
end;
subplot(211); xlabel('Frequency [Hz]')
loglog(w,abs(G)); title('Bode diagram - Gain');
subplot(212); xlabel('Frequency [Hz]')
semilogx(w,angle(G)); title('Bode diagram - Phase');
```

Listing 12.1 Frequency response of the continuous-time transfer function $G(s)$.

will provide estimates of

$$\theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{12.9}$$

12.3 We chose Matlab^{TM} as a tool for implementing the requested frequency response

In addition, if it is desireable to plot the errors

12.4 We consider the equations of rigid-body mechanics in the form of Euler-Lagrange eqautions

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = \tau, \qquad \text{where} \quad L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - U(q) \tag{12.10}$$

where the partial derivatives are

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}} &= M(q)\dot{q} && (12.11)\\ \frac{\partial L}{\partial q} &= \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}^T M(q)\dot{q}) - \frac{\partial \mathcal{U}}{\partial q} = \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}^T M(q)\dot{q}) - G(q) && (12.12) \end{aligned}$$

Expressed in term of the inertia forces $M(q)\ddot{q}$, the Coriolis and centripetal forces $C(q,\dot{q})\dot{q}$, the gravitation forces $G(q)$ and the applied forces τ, we have the force equation

$$\begin{aligned} \tau &= M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) && (12.13)\\ &= M(q)\ddot{q} + (\dot{M}(q,\dot{q})\dot{q} - \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}^T M(q)\dot{q})) + G(q) && (12.14) \end{aligned}$$

```
i=sqrt(-1);
b=theta(2);
a=theta(1);
Nplot=64;
w=logspace(-2.5,0.5,Nplot)';
for k=1:Nplot,
  G(k,1)=b/(i*w(k) + a);
end;
%
% Estimate the error in the transfer function
%
% (epsilon is the sequence of residuals sampled with
% sampling period h)
%
[w,fu]=DFT(u,h);
[w,fepsilon]=DFT(epsilon,h);
dG=fepsilon./fu;
%
% Bode diagram
%
subplot(211); xlabel('Frequency [Hz]')
loglog(w,abs(G),'-',w,abs(G.*dG),':',w,abs(G./dG),':');
title('Bode diagram - Gain');
subplot(212); xlabel('Frequency [Hz]')
semilogx(w,angle(G),'-',w,angle(G+dG),':',w,angle(G-dG),':');
title('Bode diagram - Phase');
```

Listing 12.2 Frequency response of the continuous-time transfer function $G(s)$.

Application of the linear operator

$$\lambda(p) = \frac{1}{1+\tau p}$$

gives the equation

$$\frac{d}{dt}(\lambda\{\frac{\partial L}{\partial \dot{q}}\}) = \lambda\{\frac{\partial L}{\partial q}\} + \lambda\{\tau\}$$

From the operator algebra we find that

$$p\lambda(p) = \frac{p}{1+\tau_0 p} = \frac{1}{\tau_0}(1 - \frac{1}{1+\tau_0 p}) = \frac{1}{\tau_0}(1-\lambda)$$

Thus we find that

$$\frac{d}{dt}(\lambda\{\frac{\partial L}{\partial \dot{q}}\}) = \frac{1}{\tau_0}\frac{\partial L}{\partial \dot{q}} - \frac{1}{\tau_0}\lambda\{\frac{\partial L}{\partial \dot{q}}\}$$

and we verify that

$$\frac{1}{\tau_0}\frac{\partial L}{\partial \dot{q}} = \frac{1}{\tau_0}\lambda\{\frac{\partial L}{\partial \dot{q}}\} + \lambda\{\frac{\partial L}{\partial q}\} + \lambda\{\tau\}$$

A kind of input-estimation algorithm can thus be suggested as follows

$$\lambda\{\tau\} = \frac{1}{\tau_0}\frac{\partial L}{\partial \dot{q}} - \frac{1}{\tau_0}\lambda\{\frac{\partial L}{\partial \dot{q}}\} - \lambda\{\frac{\partial L}{\partial q}\} = \phi\theta, \qquad \phi \in \mathbb{R}^{n\times p}, \quad \theta \in \mathbb{R}^p$$

where θ contains the unknown parameters of $L(q, \dot{q})$.

13

Bibliography

[1] R. Johansson, *System Modeling and Identification*, Prentice Hall, Englewood Cliffs, NJ, 1993.