LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN10 Multivariable Control

Exam 2020-01-08, 08:00–13:00

Points and grades

All solutions must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem. *Preliminary* grade limits:

- Grade 3: 12 points
- Grade 4: 17 points
- Grade 5: 22 points

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized "Formelsamling i Reglerteknik"/"Collection of Formulas" and a pocket calculator. Handouts of lecture notes and lecture slides (including markings/notes) are also allowed.

Results

The result of the exam will be entered into LADOK. The solutions will be available on the course home page: `http://www.control.lth.se/course/FRTN10`

Solutions to Exam in FRTN10 Multivariable Control 2020-01-08

1. Consider the following MIMO linear system:

$$G(s) = \frac{1}{s(s+1)(s+2)} \begin{pmatrix} s(s+2) & 0 & s^2 \\ -(s+1)(s+2) & s(s+1) & s(s+1) \end{pmatrix}$$

a. How many inputs and outputs does the system have? (0.5 p)

b. Determine the poles and (transmission) zeros of the system, including their multiplicity. (2 p)

c. What is the L_2 gain of the system? (0.5 p)

Solution

a. The dimensions of G gives that the system has three inputs and two outputs.

b. The relevant subdeterminants of order 1 are the five non-zero elements

$$\frac{1}{s+1}, \quad \frac{s}{(s+1)(s+2)}, \quad \frac{-1}{s}, \quad \frac{1}{(s+2)}, \quad \frac{1}{(s+2)}$$

and the 3 subdeterminants of order 2, corresponding to deletion of the respective columns, are

$$\frac{-s}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \frac{1}{(s+1)(s+2)}.$$

Considering all subdeterminants, we see that the least common denominator is

$$p(s) = s(s+1)(s+2)^2.$$

The system has therefore four poles: one at $s = 0$, one at $s = -1$ and two at $s = -2$.

To determine the zeros of the system, adjust the subdeterminants of order two so that their denominators are the pole polynomial $p(s)$. We get

$$\frac{-s^2}{p(s)}, \quad \frac{2s(s+2)}{p(s)}, \quad \frac{s(s+2)}{p(s)}.$$

The common divisor for these subdeterminants is the zero polynomial $z(s) = s$. Thus, the system has a single RHP zero located at $s = 0$.

c. The system has an integrator; subsystem $(2,1)$ is $-1/s$. The system thus has infinite (or undefined) L_2 gain.

2. Consider the system depicted in Figure 1.

a. Use the Small Gain Theorem to determine a stability bound for $\|\Delta\|$. (1 p)

b. In the case where $\Delta = -K$, with $K > 0$ constant, determine the stability bound for K. (1 p)

c. Explain the difference between the two bounds obtained in **a** and **b**. (1 p)

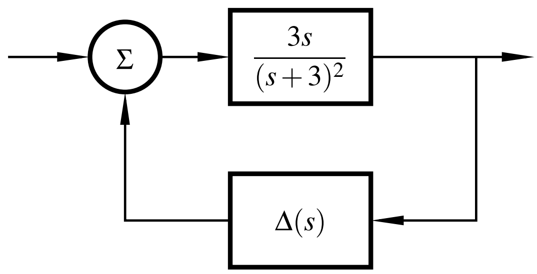

Figure 1: Block diagram for problem 2.

Solution

a. The Small Gain Theorem says that

$$\left\| \frac{3s}{(s+3)^2} \right\| \cdot \|\Delta\| < 1$$

is sufficient to guarantee stability. The maximum gain of $\frac{3s}{(s+3)^2}$ can be obtained by sketching its Bode magnitude diagram or by calculating

$$\sup_\omega \frac{|3i\omega|}{|i\omega+3|^2} = \sup_\omega \frac{3\omega}{\omega^2+9} = [\omega \to 3] = \frac{1}{2}$$

It can thus be concluded that $\|\Delta\| < 2$ for stability.

b. With $\Delta = -K$, the closed-loop poles can be determined directly. The characteristic equation becomes

$$(s+3)^2 + K3s = 0$$

which has stable (LHP) roots for all $K > 0$.

c. The SGT is conservative, no assumptions are made on Δ (sign, for example). In **b)**, the poles can be determined directly, which tells us exactly when the system is stable.

3. Your boss has heard that you are great at automatic control and wants some help with finding a good controller for the process $P(s) = \frac{1}{1+s}e^{-0.5s}$.

a. He wants a fast system, the closed-loop bandwidth ω_b should be at least 5 rad/s. Will you be able to satisfy this specification? (0.5 p)

b. Your boss gets a little impatient and tries to find a controller himself. He claims that he will get a fast enough system that has the crossover frequency $\omega_c = 5$ rad/s with the controller

$$C(s) = \frac{K(1+s/5)}{s/5}e^{0.5s}$$

Find out what K he used to get this ω_c. (1 p)

c. Your boss is happy and shows you the margin plot of the open loop system PC where everything looks nice, see Figure 2. However, you see a very big problem with his design. What is the problem that you need to explain to your boss? (0.5 p)

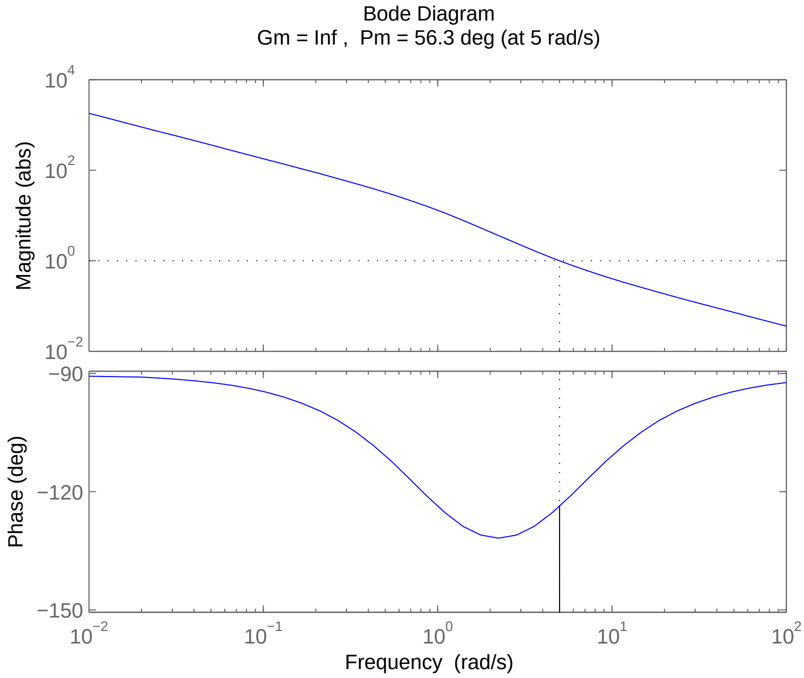

Figure 2: Margin plot for the open loop system in problem 3b.

d. You decide that an Internal Model Controller is a better choice here. Design such a controller for the process, and aim for a closed-loop bandwidth of approximately 1 rad/s. (3 p)

Solution

a. No, you will not be able to fulfill the specification due to the fundamental limitation on the bandwidth of a time-delayed system. A system on the form $G(s) = G_1(s)e^{-sL}$ will have the limited bandwidth $\omega_B < 1/L$ which in this case means that you can get a bandwidth of at most 2 rad/s.

b. The crossover frequency is given as $|PC(i\omega_c)| = 1$.

$$PC(s) = \frac{K(1+s/5)}{s/5(1+s)}$$

with $\omega_c = 5$ we get

$$|PC(5i)| = \frac{K|1+i|}{|i||1+5i|} = \frac{K\sqrt{1^2+1^2}}{1\sqrt{1^2+5^2}} = \frac{K\sqrt{2}}{\sqrt{26}} \Rightarrow K = \sqrt{13}.$$

c. You can not implement a controller with the factor $e^{0.5s}$ in it, since it is not causal.

d. By removing the time-delay in the IMC design, and compensating for the fact that the process is strictly proper, we get

$$Q(s) = \frac{s+1}{\lambda s+1}.$$

The closed-loop system is then given by

$$P(s)Q(s) = \frac{1}{\lambda s + 1} e^{-0.5s}.$$

If we pick $\lambda = 1$ we get roughly a closed-loop bandwidth of 1 rad/s. Thus, the final IMC controller is

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{s+1}{s+1-e^{-0.5s}}.$$

4. Consider an integrator process driven by unit intensity white noise:

$$\dot{x}(t) = w_1(t), \qquad R_1 = 1$$

a. Assume that there is one noisy measurement signal, given by

$$y(t) = x(t) + w_2(t), \qquad R_2 = 1$$

Assuming an optimal Kalman filter, compute the minimum observer error variance. (1 p)

b. Assume that there are two independent noisy measurements, given by

$$\begin{aligned} y_1(t) &= x(t) + w_{21}(t), \qquad & R_{21} = r \\ y_2(t) &= x(t) + w_{22}(t), \qquad & R_{22} = r \end{aligned}$$

where $r > 0$ is the intensity of both measurement noise processes. Assuming an optimal Kalman filter, for what values of r will the observer error variance be smaller than in **a**? (2 p)

c. Assume that the process noise w_1 noise is no longer white but a zero-mean stationary random process with spectrum

$$\Phi_{w_1}(\omega) = \frac{9 + 36\omega^2}{(1 + 4\omega^2)^2}$$

Find a state-space description of the noise-generating system and extend the original state-space model with the noise system. Let the new input be a unit intensity white noise process v. (2 p)

Solution

In both cases, we are looking for the observer error covariance $\mathrm{E}\tilde{x}^2 = P$, where P is given by the solution to the algebraic Riccati equation

$$AP + PA^T + R_1 - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T = 0$$

a. In this case we have $A = 0$, $C = 1$, $R_1 = R_2 = 1$, $R_{12} = 0$ and the Riccati equation becomes

$$1 - P^2 = 0$$

with the solution $P = 1$.

b. In this case we have $A = 0$, $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $R_1 = 1$, $R_2 = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, $R_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the Riccati equation becomes

$$1 - \tfrac{2}{r}P^2 = 0$$

with the solution $P = \sqrt{\frac{r}{2}}$. For this to be smaller than in **a**, we must have $r < 2$.

c. First we do a spectral factorization of $\Phi_{w_1}(\omega)$:

$$\Phi_{w_1}(\omega) = \frac{9+36\omega^2}{(1+4\omega^2)^2} = \frac{9}{(1+4\omega^2)} = \frac{3}{(1+2\omega i)} \cdot \frac{3}{(1-2\omega i)}.$$

Thus we have the noise model

$$w_1(s) = \frac{3}{1+2s} v(s),$$

which has the state-space realization

$$\dot{w}_1 = -\frac{1}{2} w_1 + \frac{3}{2} v.$$

The extended state space model is the given by

$$\begin{bmatrix} \dot{x} \\ \dot{w}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ w_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3/2 \end{bmatrix} v.$$

5. Consider the block diagram of a cascaded controller structure in Fig. 3, where C_1, C_2 represent controllers and P_1, P_2 represent subsystems of the plant we wish to control. The signals in the system are

- z_1, z_2 : output signals from P_1 and P_2 respectively.
- r_1, r_2 : reference signals for z_1 and z_2 respectively.
- d_1, d_2 : load disturbances on the inputs to P_1 and P_2 respectively.
- n_1, n_2 : measurement noise on the outputs from P_1 and P_2 respectively.

All signals are assumed to be multivariate.

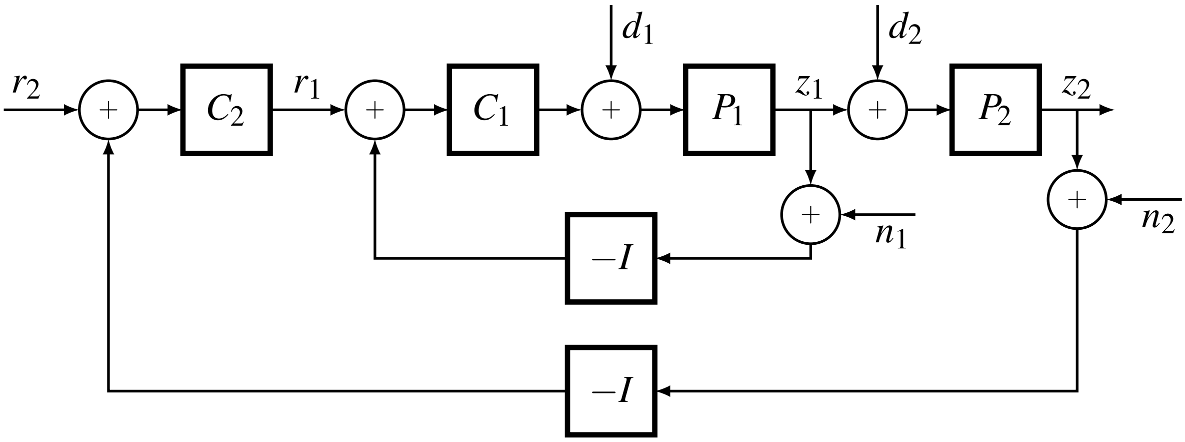

Figure 3: Block diagram considered in Problem 5

a. Let $w_1 = [r_1\, d_1\, n_1]^\intercal$. The signal z_1 is then given by the relation

$$z_1 = G_{z_1 w_1} w_1.$$

Give the expression for $G_{z_1 w_1}$. You are allowed to use C_1, C_2, P_1, P_2 in your expression. (2 p)

b. Let $w = [r_2\, d_1\, d_2\, n_1\, n_2]^\intercal$. The signal z_2 is then given by

$$z_2 = G_{z_2 w} w.$$

Give the expression for $G_{z_2 w}$. You are allowed to use C_1, C_2, P_1, P_2 and references to elements of $G_{z_1 w_1}$ from **a.** in your expression. (2 p)

Solution

a. From the block diagram in Fig. 3 we have

$$\begin{aligned} z_1 &= P_1(d_1 + C_1(r_1 - (n_1 + z_1))), \\ &\Leftrightarrow \\ z_1 &= (I + P_1 C_1)^{-1} P_1 (C_1 r_1 + d_1 - C_1 n_1). \end{aligned}$$

Thus we see that $G_{z_1 w_1}$ is given by:

$$G_{z_1 w_1} = (I + P_1 C_1)^{-1} P_1 \left[C_1, \quad I, \quad -C_1 \right].$$

b. Let $G_{z_1w_1} = [G_{z_1r_1} \quad G_{z_1d_1} \quad G_{z_1n_1}]$ where the elements are given by the expression from **a.**. From the block diagram in Fig. 3 we then have

$$\begin{aligned}
z_2 &= P_2(z_1 + d_2), \\
z_1 &= G_{z_1r_1} r_1 + G_{z_1d_1} d_1 + G_{z_1n_1} n_1, \\
r_1 &= C_2(r_2 - (z_2 + n_2)), \\
&\Leftrightarrow \\
z_2 &= P_2(G_{z_1r_1} C_2(r_2 - (z_2 + n_2) + G_{z_1d_1} d_1 + d_2 + G_{z_1n_1} n_1), \\
&\Leftrightarrow \\
z_2 &= (I + P_2 G_{z_1r_1} C_2)^{-1} P_2 (G_{z_1r_1} C_2(r_2 - n_2) + G_{z_1d_1} d_1 + d_2 + G_{z_1n_1} n_1).
\end{aligned}$$

Thus we see that G_{z_2w} is given by:

$$G_{z_2w} = (I + P_2 G_{z_1r_1} C_2)^{-1} P_2 [G_{z_1r_1} C_2, \quad G_{z_1d_1}, \quad I, \quad G_{z_1n_1}, \quad -G_{z_1r_1} C_2].$$

6. Recall the loop-shaping design of Lab 1. Your clever classmate has the idea to control the second mass using an LQG controller instead. The system augmented with Gaussian additive noise is described by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + w_1(t) \\
y(t) &= Cx(t) + w_2(t)
\end{aligned}$$

where w_1 and w_2 are uncorrelated white noise with covariances $R_1 = I_{4\times4}$ and $R_2 = 0.0025$ respectively, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{d_1}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & -\frac{d_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k_m}{m_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 0 \quad k_y \quad 0].$$

We construct the objective function as

$$J = \int_0^\infty \left(x^\top(t) Q_1 x(t) + u^2(t) \right) dt.$$

After some tuning you arrive at

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/100 \end{bmatrix}, \qquad Q_2 = 20;$$

you manage to fulfill all the design specifications except the disturbance rejection, see Figure 4. The specifications are:

- Well-damped.
- Rise-time between 0.2 and 0.6 seconds.

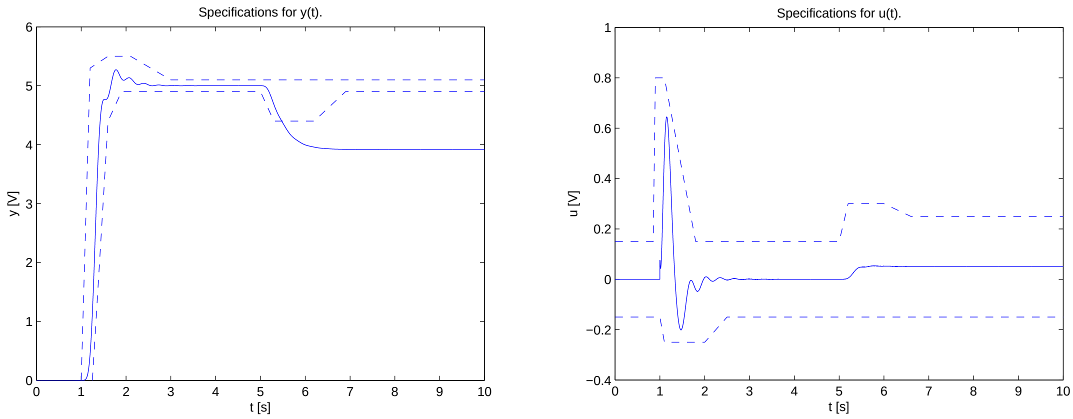

Figure 4: The output y (left) and control signal u (right) when applying the LQG regulator in Problem 6 after a reference step followed by a step disturbance.

- Settling time ≤ 2 seconds.
- Rejection of constant load disturbances in at most 2 seconds.

The Gang-of-Four plot for the controlled system is shown in Figure 5.

a. Looking at the maximum sensitivity and complementary sensitivity, is the system reasonably robust? (1 p)

b. Why does the controller fail to reject a constant load disturbance? How can you deduce this behavior from the Gang-of-Four plot in Figure 5? (1 p)

c. Modify the system such that an LQG-controller designed for such a system will reject constant disturbances. You are expected to explicitly write down the matrices which constitute the parameters of the Riccati equations you would need to solve. If you need to change the weight- or covariance matrices, give an explanation for doing so and provide reasonable guesses for any parameters you add. (2 p)

d. The modification in **c.** will likely result in a controller which quickly rejects the load disturbance, but has a large overshoot during the reference step. If this is the case, what could you do to solve this problem? Give at least two suggestions. (1 p)

Solution

a. From the Gang of Four we read off $M_t = 1$ and $M_s \approx 2$, which could be considered reasonably robust.

b. The process fails to reject a constant load disturbance because it is lacking integral action. This can be seen in $\dfrac{P}{1+PC}$ where the static gain is not zero.

c. We could add the integral of the control error as a state in the process model, but then the process model would no longer be stabilizable and the conditions for applying LQG design are violated. Thus we have to push the pole just into the left half-plane, i.e. pick a small positive $\delta \approx 0$.

$$A_e = \begin{bmatrix} A & 0 \\ C & -\delta \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0 \end{bmatrix}$$

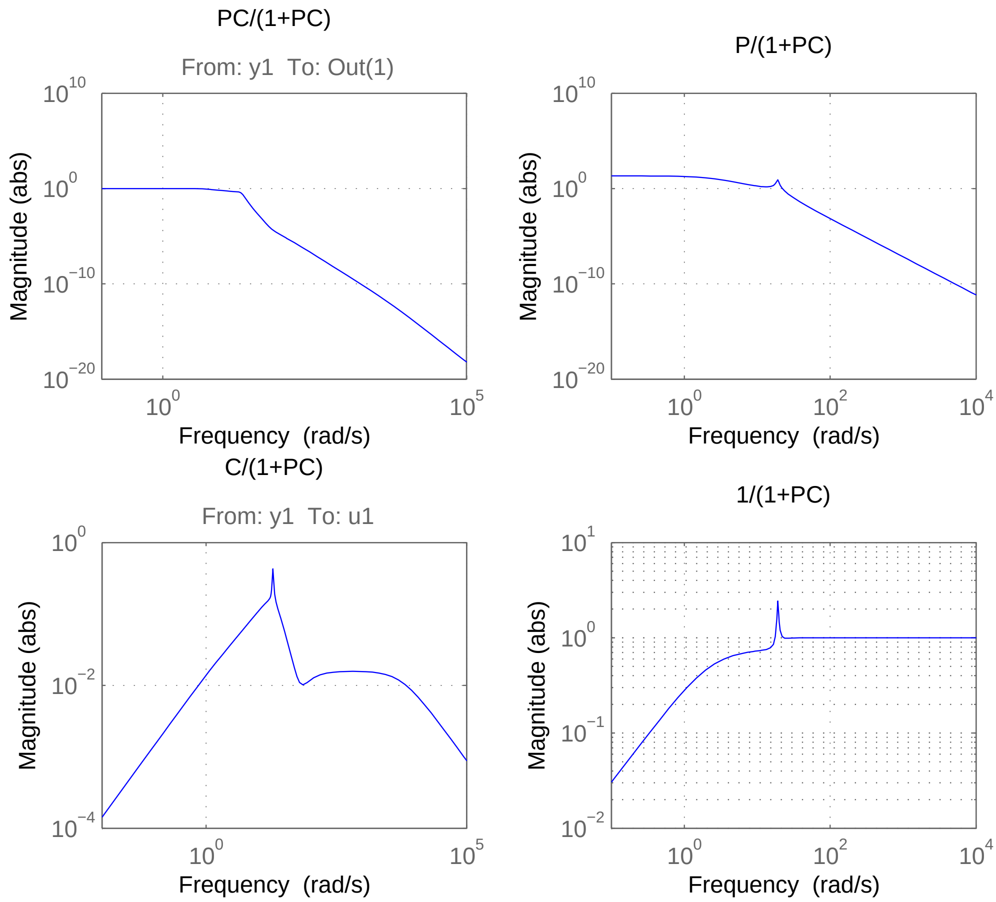

Figure 5: Gang-of-Four plot for Problem 6

Then we need to augument $R_1^e = I_{5\times 5}$ and

$$Q_1^e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/100 & 0 \\ 0 & 0 & 0 & 0 & q_i \end{bmatrix},$$

where q_i should not be too large, as it will make the controller very aggressive. Q_2^e and R_2^e may be kept as is.

d. Reasonable things to try is lower penalty on the *integral* state, increase the penalty on the input or x_4. We can also try adding a low-pass filter as feedforward control.