LUNDS
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Lecture 9

FRTN10 Multivariable Control

Automatic Control LTH, 2019

Course Outline

L1–L5 Specifications, models and loop-shaping by hand

L6–L8 Limitations on achievable performance

L9–L11 Controller optimization: analytic approach

- 9 **Linear-quadratic control**
- 10 Kalman filtering
- 11 LQG control

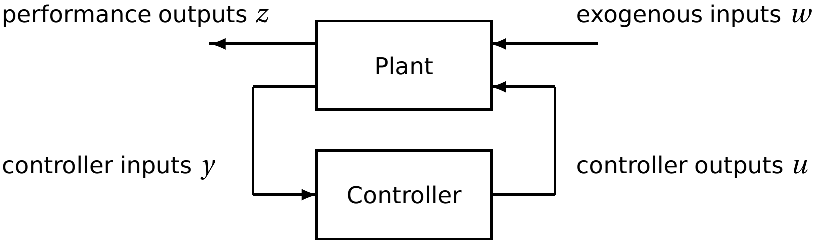
L12–L14 Controller optimization: numerical approach

L15 Course review

L9: Linear-quadratic control

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

A general optimization setup

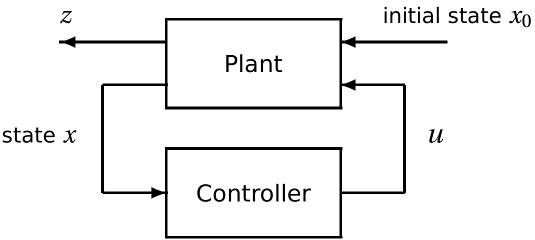


General objective: find a controller that optimizes the closed-loop system $G_{zw}(s)$.

Lectures 9–11: Problems with analytic solutions

Lectures 12–14: Problems with numeric solutions

Today's problem: Optimal state feedback



Optimization problem:

$$\min_{u[0,\infty)} J = \int_0^\infty |z|^2 \, dt = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$

$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$ is a symmetric, positive semidefinite matrix and $Q_2 > 0$

Why linear-quadratic control?

- Simple, analytic solution
  - Quadratic cost functions give linear state feedback control laws
- Always stabilizing
- Works for MIMO systems
- Guaranteed robustness (in the state feedback case)
- Foundation for more advanced methods like model-predictive control (MPC)

L9: Linear-quadratic control

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

Dynamic programming: simple example

Determine u_0 and u_1 if the objective is to minimize

$$x_1^2 + x_2^2 + u_0^2 + u_1^2$$

when

$$x_1 = x_0 + u_0$$
$$x_2 = x_1 + u_1$$

Solution: Start at the last stage and proceed backwards to solve the problem sequentially:

- First find optimal u_1 as function of x_1
- Then find optimal u_0 as function of x_0

Dynamic programming: simple example

Break the problem into smaller parts that can be solved sequentially:

$$\min_{u_0,u_1}\left\{x_1^2+x_2^2+u_0^2+u_1^2\right\}=\min_{u_0}\left\{x_1^2+u_0^2+\underbrace{\min_{u_1}\left\{x_2^2+u_1^2\right\}(x_1)}_{J_1(x_1)}\right\}$$

Dynamic programming: simple example

Break the problem into smaller parts that can be solved sequentially:

$$\min_{u_0,u_1}\left\{x_1^2+x_2^2+u_0^2+u_1^2\right\}=\min_{u_0}\left\{x_1^2+u_0^2+\underbrace{\min_{u_1}\left\{x_2^2+u_1^2\right\}(x_1)}_{J_1(x_1)}\right\}$$

Minimize by completion of squares:

$$J_1(x_1)=\min_{u_1}\left\{(x_1+u_1)^2+u_1^2\right\}=\min_{u_1}\left\{2\left(u_1+\tfrac{1}{2}x_1\right)^2+\tfrac{1}{2}x_1^2\right\}$$

$$=\tfrac{1}{2}x_1^2 \quad \text{with minimum attained for } u_1=-\tfrac{1}{2}x_1$$

$$J_0(x_0)=\min_{u_0}\left\{(x_0+u_0)^2+u_0^2+J_1(x_0+u_0)\right\}=\min_{u_0}\left\{\tfrac{5}{2}\left(u_0+\tfrac{3}{5}x_0\right)^2+\tfrac{3}{5}x_0^2\right\}$$

$$=\tfrac{3}{5}x_0^2 \quad \text{with minimum attained for } u_0=-\tfrac{3}{5}x_0$$

Quadratic optimal cost

It can be shown that the optimal cost on a time interval $[t, \infty)$ is quadratic:

$$\min_{u[t,\infty)} \int_t^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau = x^T(t) S x(t), \quad S = S^T > 0$$

when

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \geq 0, \quad Q_2 > 0$$

Dynamic programming, Richard E. Bellman, 1957

Bellman's principle of optimality: An optimal trajectory on the time interval $[t, T]$ must be optimal also on each of the subintervals $[t, t+\epsilon]$ and $[t+\epsilon, T]$.

Dynamic programming for linear-quadratic control

For an infinitesimal time step of length ϵ,

$$x(t+\epsilon) = x(t) + (Ax(t) + Bu(t))\epsilon \qquad \text{as } \epsilon \to 0$$

Dynamic programming for linear-quadratic control

For an infinitesimal time step of length ϵ,

$$x(t+\epsilon) = x(t) + (Ax(t) + Bu(t))\epsilon \qquad \text{as } \epsilon \to 0$$

Invoking the principle of optimality for $[t, t+\epsilon]$ and $[t+\epsilon, \infty]$:

$$x^T(t)Sx(t) = \min_{u[t,\infty)} \int_t^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau$$

$$= \min_{u[t,\infty)} \left\{ \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T Q \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \epsilon + \int_{t+\epsilon}^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau \right\}$$

$$= \min_{u(t)} \Big\{ \big(x^T(t)Q_1x(t) + 2x^T(t)Q_{12}u(t) + u^T(t)Q_2u(t) \big)\epsilon$$
$$+ \Big[x(t) + (Ax(t) + Bu(t))\epsilon \Big]^T S \Big[x(t) + (Ax(t) + Bu(t))\epsilon \Big] \Big\}$$

Dynamic programming for linear-quadratic control

Neglecting the ϵ^2 terms gives **Bellman's equation**:

$$0 = \min_{u(t)} \Big\{ x^T(t) Q_1 x(t) + 2x^T(t) Q_{12} u(t) + u^T(t) Q_2 u(t) \Big) \\ + 2x^T(t) S \Big(Ax(t) + Bu(t) \Big) \Big\}$$

L9: Linear-quadratic control

Completion of squares – matrix case

Suppose $Q_u > 0$. Then the quadratic form

$$x^T Q_x x + 2x^T Q_{xu} u + u^T Q_u u$$

$$= (u + Q_u^{-1} Q_{xu}^T x)^T Q_u (u + Q_u^{-1} Q_{xu}^T x) + x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x$$

is minimized w.r.t. u by

$$u = -Q_u^{-1} Q_{xu}^T x$$

The minimum is

$$x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x$$

The Riccati equation

Completion of squares in Bellman's equation gives

$$
\begin{aligned}
0 &= \min_{u_t} \left\{ \left(x_t^T Q_1 x_t + 2x_t^T Q_{12} u_t + u_t^T Q_2 u_t \right) + 2x_t^T S \left(Ax_t + Bu_t \right) \right\} \\
&= \min_{u_t} \left\{ x_t^T [Q_1 + A^T S + SA] x_t + 2x_t^T [Q_{12} + SB] u_t + u_t^T Q_2 u_t \right\} \\
&= x_t^T \left(Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T \right) x_t
\end{aligned}
$$

with minimum attained for

$$
u_t = -Q_2^{-1} (SB + Q_{12})^T x_t
$$

The Riccati equation

Completion of squares in Bellman's equation gives

$$
\begin{aligned}
0 &= \min_{u_t} \left\{ \left(x_t^T Q_1 x_t + 2 x_t^T Q_{12} u_t + u_t^T Q_2 u_t \right) + 2 x_t^T S \left(A x_t + B u_t \right) \right\} \\
&= \min_{u_t} \left\{ x_t^T [Q_1 + A^T S + SA] x_t + 2 x_t^T [Q_{12} + SB] u_t + u_t^T Q_2 u_t \right\} \\
&= x_t^T \left(Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T \right) x_t
\end{aligned}
$$

with minimum attained for

$$u_t = -Q_2^{-1} (SB + Q_{12})^T x_t$$

The equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

is called the **algebraic Riccati equation**

Algebraic Riccati equations in Matlab

```
icare  Implicit solver for continuous-time Riccati equations.

  [X,K,L] = icare(A,B,Q,R,S,E,G) computes the stabilizing solution X of
  the continuous-time algebraic Riccati equation
                                                 -1
     A'XE + E'XA + E'XGXE - (E'XB + S) R  (B'XE + S') + Q = 0 .

  The matrices Q,R,G must be Hermitian and R,E must be invertible. When
  omitted or set to [], B,R,S,E,G default to the values B=0, R=I, S=0,
  E=I, and G=0. Scalar-valued Q,R,G are interpreted as multiples of the
  identity matrix. icare also returns the state-feedback gain K and the
  closed-loop eigenvalues L given by
          -1
     K = R  (B'XE + S'),    L = EIG(A+G*X*E-B*K,E) .

icare returns [] for X,K when there is no finite stabilizing solution.
```

(Note: In older versions of Matlab the command is called care)

L9: Linear-quadratic control

Stabilizability

A system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is called **stabilizable** if its uncontrollable subspace is stable.

Controllability $\Rightarrow$ stabilizability

Linear-quadratic control – summary

Control problem:

Minimize $$J = \int_0^\infty \Big(x^T(t) Q_1 x(t) + 2x^T(t) Q_{12} u(t) + u^T(t) Q_2 u(t) \Big) dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$

Solution: Assuming a stabilizable system, there exists a unique $S = S^T > 0$ solving the algebraic Riccati equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

The optimal control law is $u = -Lx$ with $L = Q_2^{-1}(SB + Q_{12})^T$.

The optimal cost is $J^* = x_0^T S x_0$.

Remarks

Note that the optimal control law does not depend on x_0.

The optimal feedback gain L is static since we are solving an infinite-horizon problem.

(LQ theory can also be applied to finite-horizon problems and to problems with time-varying system matrices. We then obtain a Riccati differential equation for $S(t)$ and a time-varying state feedback, $u(t) = -L(t)x(t)$)

Example: Control of an integrator

For $\dot{x}(t) = u(t)$, $x(0) = x_0$,

Minimize $$J = \int_0^\infty \left\{ x(t)^2 + \rho u(t)^2 \right\} dt$$

Riccati equation $$0 = 1 - S^2/\rho \quad \Rightarrow \quad S = \sqrt{\rho}$$

Controller $$L = S/\rho = 1/\sqrt{\rho} \quad \Rightarrow \quad u = -x/\sqrt{\rho}$$

Closed loop system $$\dot{x} = -x/\sqrt{\rho} \quad \Rightarrow \quad x = x_0 e^{-t/\sqrt{\rho}}$$

Optimal cost $$J^* = x_0^T S x_0 = x_0^2 \sqrt{\rho}$$

What values of ρ give the fastest response? Why?

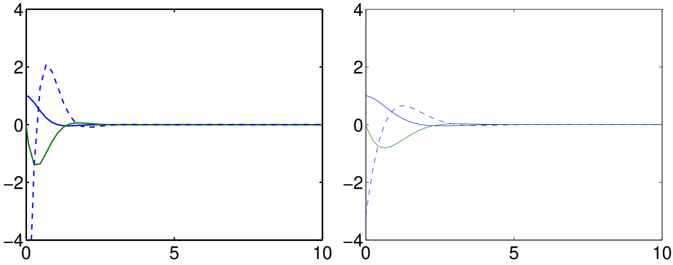
Solving the LQ problem in Matlab

```
lqr  Linear-quadratic regulator design for state space systems

[K,S,E] = lqr(SYS,Q,R,N) calculates the optimal gain matrix K
such that:

* For a continuous-time state-space model SYS, the state-
feedback law u = -Kx  minimizes the cost function

J = Integral {x'Qx + u'Ru + 2*x'Nu} dt

subject to the system dynamics  dx/dt = Ax + Bu

The matrix N is set to zero when omitted. Also returned are
the solution S of the associated algebraic Riccati equation
and the closed-loop eigenvalues E = EIG(A-B*K).
```

Example – Double integrator

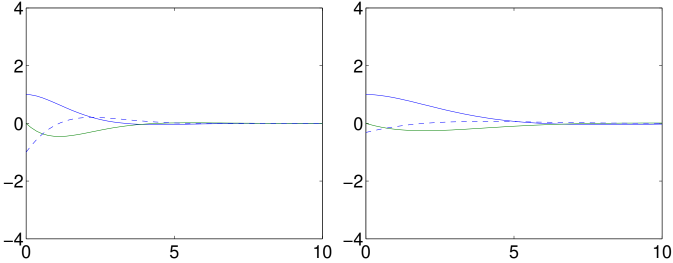
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Q_2 = \rho \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

States (full) and input (dotted) for $\rho = 0.01$, $\rho = 0.1$:



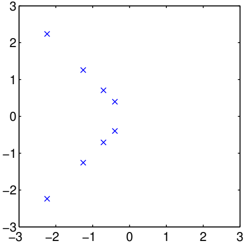
Example – Double integrator

States (full) and inputs (dotted) for $\rho = 1$, $\rho = 10$:

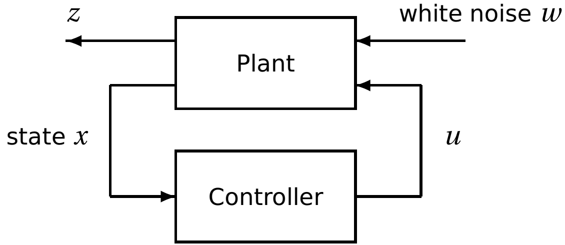


Closed loop poles:

$$s = 2^{-1/2}\rho^{-1/4}(-1 \pm i)$$



Stochastic linear-quadratic control



Minimize $\quad J = \mathrm{E}|z|^2 = \mathrm{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right\}$

subject to $\quad \dot{x}(t) = Ax(t) + Bu(t) + w(t)$

where w is white noise with intensity R. Same Riccati equation and solution (S, L) as in the deterministic case. The optimal cost is

$$J^* = \mathrm{E}\, x^T S x = \mathrm{tr}(SR)$$

L9: Linear-quadratic control

Stability of the closed-loop system

Assume that

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} > 0$$

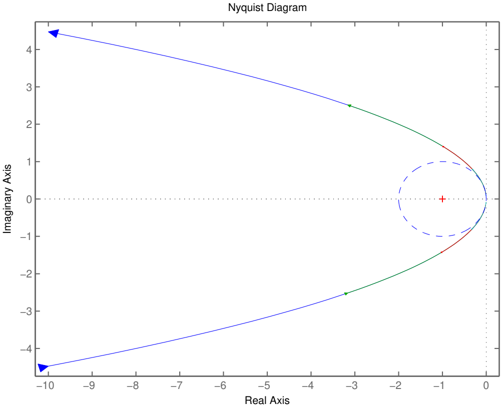
and that there exists a solution $S > 0$ to the algebraic Riccati equation. Then the optimal controller $u(t) = -Lx(t)$ gives an asymptotically stable closed-loop system $\dot{x}(t) = (A - BL)x(t)$.

Proof:

$$\begin{aligned} \frac{d}{dt} x^T(t) S x(t) &= 2x^T S \dot{x} = 2x^T S(Ax + Bu) = \text{[Bellman's equation]} \\ &= -\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \right) < 0 \text{ for } x(t) \neq 0 \end{aligned}$$

Hence $x^T(t) S x(t)$ is decreasing and tends to zero as $t \to \infty$.

Robustness of optimal state feedback



The distance from the loop gain $L(i\omega I - A)^{-1}B$ to -1 is never smaller than 1. This is always true when $Q_1 > 0$, $Q_{12} = 0$ and $Q_2 > 0$ is scalar. The phase margin is $\geq 60^\circ$ and the (positive) gain margin is infinite!

[For proof, see G&L Section 9.4]

Lecture 9 – summary

- We specify what “optimal control” means using a quadratic cost function.
- Solving an algebraic Riccati equation gives the optimal state feedback law $u = -Lx$:

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T \;\Rightarrow\; S$$
$$L = Q_2^{-1} (SB + Q_{12})^T$$

- The LQ controller has remarkable robustness properties.