Lecture 14 — Course Summary

- CEQ
- The exam
- Questions / review of the course

CEQ

You will get a mail regarding CEQ (Course evaluation) to be filled out via a web-page.

Please, fill it in, and **write your comments**.

Both Swedish and English versions are available!

Remember, without your feedback we teach in open-loop.

Question: What's on the exam?

Among old exam problems:

- Models, equilibria etc
- Linearization and stability
- Circle criterion
- Small gain
- Describing Functions
- Lyapunov functions
- Optimal Control
- ...

Old exams and solutions are available from the course home page.

Five hours written exam

Course Material Allowed:

- Lecture slides 1-14 (no exercises or old exams)
- Laboratory exercises 1, 2, and 3
- *Reglerteori* by Glad and Ljung
- *Applied Nonlinear Control* by Slotine and Li
- *Nonlinear Systems* by Khalil
- *Calculus of variations and optimal control theory* by Liberzon

You may bring everything on the list + "Collection of Formulae for Control" to the exam.

Question

Can I get different answers if I use the Small Gain theorem and the Circle criterion? What does it mean?

- If the conditions for stability are not satisfied for one criterion it does not necessarily mean that the system is unstable. It just means that you can not use that method to guarantee stability. You can never 'prove' that a system is stable with one method and 'unstable' with another.
- Similarly, there are no general guaranteed methods to find a Lyapunov function (though some suggested good methods/candidates are worth trying, e.g., quadratic, total energy, etc.).

Question

Please repeat the stability definitions and methods to prove stability.

Explain invariant sets and when $\dot{V} = 0$.

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if locally stable and

$$\lim_{t\to\infty} x(t) = 0$$

for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$ for some open set Ω. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\dot{V}(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if also

- $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $\dot{V}(x) < 0$ for all $x \neq 0$
- $\boxed{V(x) \to \infty \text{ as } \|x\| \to \infty \text{ (radial unboundedness)}}$

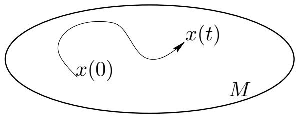
then $x = 0$ is $\boxed{\text{globally}}$ asymptotically stable.

Invariant Sets

Definition A set M is called **invariant** if for the system

$$\dot{x} = f(x),$$

$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.



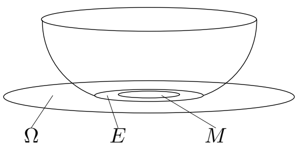
When finding Lyapunov function candidates with $\frac{dV}{dt} \leq 0$ we often want to use this to show that the origin is the largest invariant set M.

Invariant Set Theorem

Theorem Let $\Omega \in \mathbf{R}^n$ be a bounded and closed set that is invariant with respect to

$$\dot{x} = f(x).$$

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$



Example

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - x_1^3$$

Try with

$$V(x) = x_1^2 + x_2^2 \qquad \text{(Alt. 1)}$$

or

$$V(x) = 0.5x_1^4 + x_2^2 \qquad \text{(Alt. 2)}$$

Alt 1:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt}$$
$$= 2x_1x_2 + 2x_2(-x_2 - x_1^3) = 2(x_1 - x_1^3)x_2 - 2x_2^2 \qquad !??$$

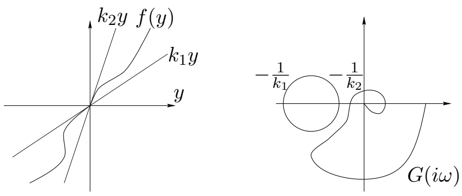
No information as indefinite.

Alt 2:

$$\frac{dV}{dt} = 0.5 \cdot 4x_1^3 x_2 + 2x_2(-x_2 - x_1^3) = -2x_2^2 \leq 0$$

To show *asymptotic* stability we need to continue (Alt.2) and use LaSalle or the invariance set theorem!

The Circle Criterion, $0 < k_1 \leq k_2 < \infty$



Theorem Consider a feedback loop with $y = Gu$ and $u = -f(y)$. Assume $G(s)$ is stable and that

$$k_1 \leq \frac{f(y)}{y} \leq k_2.$$

If the Nyquist curve of $G(s)$ stays outside the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

The other cases

G: **stable system**

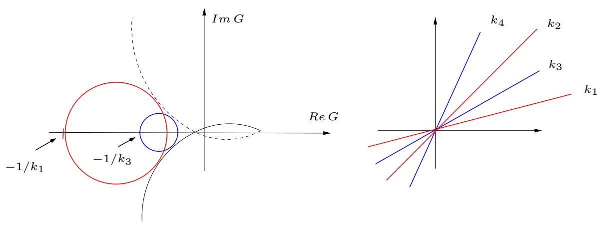
- $0 < k_1 < k_2$: Stay outside circle
- $0 = k_1 < k_2$: Stay to the right of the line Re $s = -1/k_2$
- $k_1 < 0 < k_2$: Stay inside the circle

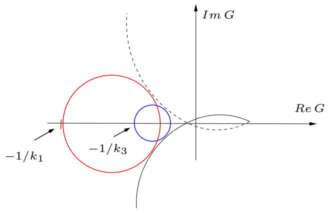
Other cases: Multiply f and G with -1.

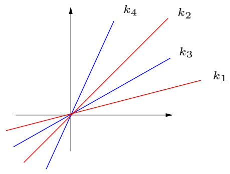
Circle criterion / Sector conditions

What does it mean that we can get different sectors when using the circle criterion for a nonlinearity in feedback with a (fixed) linear system?

Can I have many different sector conditions, and what does that mean?

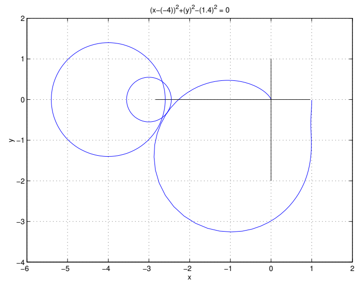




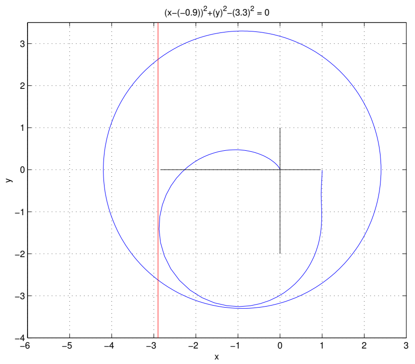


In the example above, the circle criterion can guarantee absolute stability for a nonlinearity which is bounded to **either** the sector $[k_1, k_2]$ **or** $[k_3, k_4]$ or in many other sectors, but NOT for a nonlinearity which is allowed to have a full variation within the sector $[k_1, k_4]$.

Example: $G(s) = \dfrac{1000}{(s+10)(s^2+2s+100)}$
in negative feedback with a sector bounded nonlinearity.



sector [0.28, 0.41] or [0.19, 0.38]



sector [0, 1/2.9] or [-0.41, 0.23]

Small gain theorem will give symmetric sector [-0.27, 0.27] as $||G|| = 3.69$.

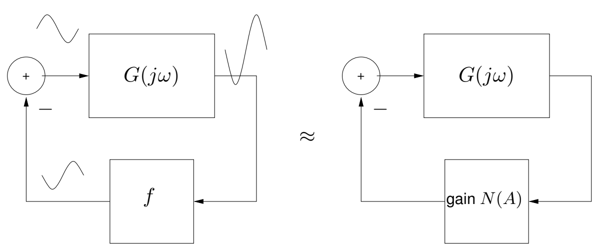
Questions

Is it possible to draw phase portraits for systems of order higher than two?

Can the describing function method be improved by including more coefficients from the Fourier series expansion?

Are there criteria to verify the low-pass character needed in a describing function argument?

Idea of Describing Function



Find frequency ω and amplitude A such that

$$G(j\omega) \cdot N(A) = -1$$

Idea of Describing Function

$$e(t) = A \sin \omega t = \text{Im}\,(Ae^{i\omega t})$$

$e(t) \to \boxed{\text{N.L.}} \to u(t)$

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

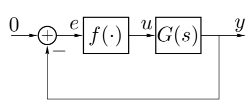
$e(t) \to \boxed{N(A,\omega)} \to u_1(t)$

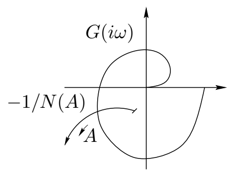
$$\begin{aligned} u_1(t) &= a_1 \cos(\omega t) + b_1 \sin(\omega t) \\ &= \text{Im}\,(N(A,\omega)Ae^{i\omega t}) \end{aligned}$$

where the **describing function** is defined as

$$N(A,\omega) = \frac{b_1(\omega) + ia_1(\omega)}{A} \implies U(i\omega) \approx N(A,\omega)E(i\omega)$$

Existence of Limit Cycles



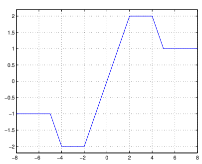


$$y = G(i\omega)u = -G(i\omega)N(A)y \quad \Rightarrow \quad G(i\omega) = -\frac{1}{N(A)}$$

The intersections of $G(i\omega)$ and $-1/N(A)$ give ω and A for *possible* limit cycles.

Harder if N is a function of both A and ω.

Example from exam 20090601 (a)



Which one of the three describing functions below corresponds to the nonlinearity $f(x)$ above?

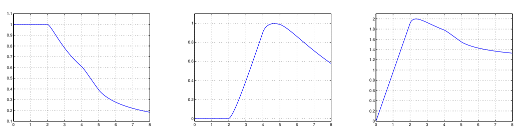
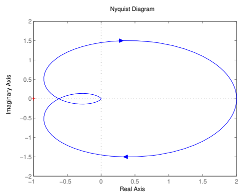
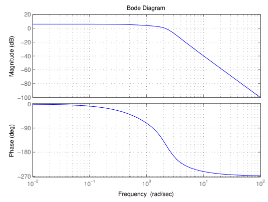


Figure: Describing functions $1 - -3$

Example from exam 20090601 (b)

Below we have the Nyquist and Bode curves of a stable linear system G. Assume that there exists non-linearities corresponding to to the three describing functions on previous page, and that each of these would be used in a negative feedback connection with G. For which do we possibly get limit cycles? If so, state possible amplitudes of the limit cycles and if they are stable or unstable?



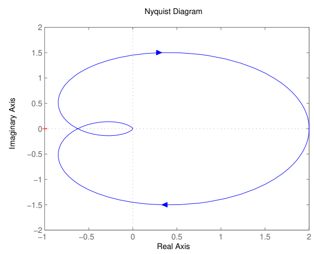


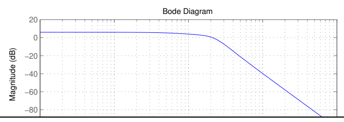
Since the third describing function fulfills that $-\frac{1}{N(2)} = -\frac{1}{2}$ and $G(i\omega_o) \approx -0.6$, we understand that we have **two** intersections. The first intersection occurs when $A \approx 1.8$ and the second intersection occurs when $A \approx 4.5$.

Examining the describing function around the first intersection, we see that $-\frac{1}{N(A)}$ goes from the outside of $G(i\omega)$ to the inside, with increasing A. Hence, we conclude that the possible limit cycle at $A \approx 1.8$ is unstable. By similar argument, we understand that the possible limit cycle at $A \approx 4.5$ is stable.

What would the corresponding frequency of the limit cycles in **(b)** be?

The frequency of all possible limit cycles is approximately 2.5 rad/s. To understand this, we see in the Bode plot that for $\omega \approx 2.5$ we have that $\arg(G(i\omega)) \approx -180$.



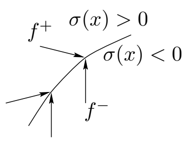


Question

Please repeat the most important facts about sliding modes.

Sliding Modes

$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$



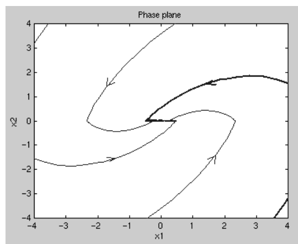
The **sliding set** is where $\sigma(x) = 0$ and f^+ and f^- point towards $\sigma(x) = 0$.

The **sliding dynamics** are $\dot{x} = \alpha f^+ + (1-\alpha)f^-$, where α is obtained from $\dot{\sigma} = \frac{\partial \sigma}{\partial x}\dot{x} = 0$.
(More precisely, find α such that the components of f^+ and f^- perpendicular to the switching surface cancel.)

Sliding Mode Dynamics

The dynamics along the sliding surface $\sigma(x) = 0$ can also be obtained by setting $u = u_{\text{eq}} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.

u_{eq} is called the **equivalent control**.



Phase plane for example in lecture 12.

Example

$$\begin{aligned} \dot{x}_1 &= 1 - u/4 \\ \dot{x}_2 &= u, \\ u &= -\text{sign}\, x_2, \quad (\text{i.e.}, \sigma(x) = x_2) \end{aligned} \tag{1}$$

What is the *sliding set* and what is *the sliding dynamics* for the system above?

If

$$\sigma(x) > 0 \Rightarrow u = -1 \Rightarrow f^+ = \begin{bmatrix} 5/4 \\ -1 \end{bmatrix}$$

$$\sigma(x) < 0 \Rightarrow u = +1 \Rightarrow f^- = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

The sliding set:

Find those values of the states at the switching curve for which

$$\nabla\sigma \cdot f^+ < 0$$

and

$$\nabla\sigma \cdot f^- > 0$$

(means that the vector fields on either side of $\sigma(x)$ points towards $\sigma(x)$, i.e., the normal projection of f^+ is negative and the normal projection of f^- on $\sigma(x)$ is positive). If these conditions are not fulfilled we will just “flow through $\sigma(x)$...”

In this example all the values along $x_2 = 0$ will belong to the switching set. (Compare with example from lecture 9 where the switching set will be restricted to $x_2 = 0$ and $-1 \leq x_1 \leq 1$, see figure on slide above).

The sliding dynamics:

Alternative 1.: Solve via normal projection on σ:

Pick α such that for $\dot{x} = \alpha f^{+} + (1-\alpha) f^{-}$, we get
$\dot{\sigma} = 0 \Rightarrow \dot{x}_2 = \alpha f_2^{+} + (1-\alpha) f_2^{-} = 0$

This gives $\alpha = 1/2$, hence $\dot{x} = \alpha f^{+} + (1-\alpha) f^{-}$ and $\dot{x}_1 = 1$ is the sliding dynamics.

Alternative 2: Solve via Equivalent control

$\dot{\sigma}(x)_{u=u_{eq}} = 0$ and $\dot{\sigma} = \dot{x}_2 = u \Rightarrow u_{eq} = 0.$
Hence $\dot{x}_1 = 1 - u_{eq}/4 = 1$ is the sliding dynamics.

Question

Please repeat optimal control with some additional example

Problem Formulation (1)

$$
\begin{aligned}
&\text{Minimize } \int_0^{t_f} L(x(t), u(t))\, dt + \phi(x(t_f)) \\
&\dot{x}(t) = f(x(t), u(t)) \\
&u(t) \in U, \quad 0 \le t \le t_f, \qquad t_f \text{ given} \\
&x(0) = x_0
\end{aligned}
$$

$x(t) \in R^n$, $u(t) \in R^m$

U control constraints

The Maximum Principle (18.2)

Introduce the **Hamiltonian**

$$H(x,u,\lambda) = L(x,u) + \underbrace{\lambda^T(t)}_{1\times n} \underbrace{f(x,u)}_{n\times 1}.$$

Suppose optimization problem (1) has a solution $u^*(t), x^*(t)$. Then the optimal solution must satisfy

$$\min_{u\in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$\dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \phi_x^T(x^*(t_f))$$

where $H_x = \frac{\partial H}{\partial x} = [\frac{\partial H}{\partial x_1} \dots \frac{\partial H}{\partial x_n}]$, $\phi_x = \frac{\partial \phi}{\partial x}$.

Problem Formulation (2)

As in (1) but with additions:

- r end constraints

$$\Psi(x(t_f)) = \begin{pmatrix} \Psi_1(x(t_f)) \\ \vdots \\ \Psi_r(x(t_f)) \end{pmatrix} = 0$$

- free end time t_f

The Maximum Principle–General Case (18.4)

Introduce the Hamiltonian

$$H(x,u,\lambda,n_0) = n_0 L(x,u) + \lambda^T(t) f(x,u)$$

Suppose optimization problem (2) has a solution $u^*(t), x^*(t)$. Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$, and a vector $\mu \in R^r$ so that $[n_0 \;\; \mu^T] \ne 0$ and

$$\min_{u\in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \le t \le t_f,$$

where

$$
\begin{aligned}
\dot{\lambda}(t) &= -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\
\lambda(t_f) &= n_0 \phi_x^T(x^*(t_f)) + \Psi_x^T(x^*(t_f))\mu
\end{aligned}
$$

Free end time t_f

If the choice of t_f is included in the optimization and/or final state constraints, then two cases: $n_0 = 1$ or $n_0 = 0$.

Also, if the choice of t_f is included in the optimization, there is an extra constraint:

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$$

Example: Optimal storage control

$$\text{Minimize } \int_0^{t_f} [u(t)e^{rt} + cx(t)] dt$$

$$\text{subject to } \begin{cases} \dot{x} = u & 0 \le u \le M \\ x(0) = 0 \\ x(t_f) \ge A \end{cases}$$

$x =$ stock size
$u =$ production rate
$r =$ production cost growth rate
$c =$ storage cost

Example: Optimal storage control I

in standard form

$$\text{Minimize } \int_0^{t_f} [cx_1(t) + u(t)x_2(t)]dt$$

$$\text{subject to } \begin{cases} \dot{x}_1 = u \quad \dot{x}_2 = rx_2 \\ x_1(0) = 0 \quad x_2(0) = 1 \\ 0 \le u \le M \\ x_1(t_f) = A \end{cases}$$

$L(u, x) = ux_2 + cx_1$ running cost

$\phi(x) = 0$ final cost

$\psi(x) = x_1$ final constraint

t_f fixed

Optimal storage control II

Hamiltonian

$$\begin{aligned} H(x, u, \lambda, n_0) &= n_0 L(x, u) + \lambda(t)^T f(x, u) \\ &= n_0(ux_2 + cx_1) + \lambda_1 u + \lambda_2 r x_2 \end{aligned}$$

Adjoint equations

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -n_0 c \qquad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -n_0 u - \lambda_2 r$$

$$\lambda_1(t_f) = n_0 \frac{\partial \Phi}{\partial x_1}(x^*(t_f)) + \mu \frac{\partial \Psi}{\partial x_1}(x^*(t_f)) = \mu$$

$$\lambda_2(t_f) = n_0 \frac{\partial \Phi}{\partial x_2}(x^*(t_f)) + \mu \frac{\partial \Psi}{\partial x_2}(x^*(t_f)) = 0$$

Should try two cases:
normal $n_0 = 1$ and $\mu \ge 0$
abnormal $n_0 = 0$ and $\mu > 0$

Optimal storage control III

Abnormal case: $n_0 = 0$ $\mu > 0$

$$\lambda_1(t) = \mu \qquad \forall 0 \le t \le t_f$$

For every $0 \le t \le t_f$

$$u^*(t) \in \underset{u}{\operatorname{argmin}}\, H(x^*, u, \lambda, 0) = \underset{u}{\operatorname{argmin}}\{\mu u\}$$

$$u^*(t) = 0 \qquad \forall 0 \le t \le t_f$$

violates constraint

$$x_1(t_f) = A$$

Optimal storage control IV

Normal case: $n_0 = 1$ $\mu \ge 0$

$$\lambda_1(t) = b - ct, \qquad b = \mu - ct_f \qquad x_2(t) = e^{rt}$$

For every $0 \le t \le t_f$

$$u^*(t) \in \underset{u}{\operatorname{argmin}}\, H(x^*, u, \lambda, 1) = \underset{u}{\operatorname{argmin}}\{u(e^{rt} + b - ct)\}$$

$$u^*(t) = \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\ 0 & \text{if } e^{rt} + b - ct > 0 \end{cases}$$

$$u^*(t) = \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\ 0 & \text{if } e^{rt} + b - ct > 0 \end{cases} = \begin{cases} M & t_1 \le t \le t_2 \\ 0 & \text{otherwise} \end{cases}$$

$x(t_f) = A$ gives that $M(t_2 - t_1) = A$. To find t_1, solve

$$\min_{0 \le s \le A/M} \left\{ \int_s^{s+A/M} M(e^{rt} + ct)dt + \int_{s+A/M}^{t_f} cAdt \right\}$$

Extra session before the exam

Extra session for questions:

- Thursday January 9 at 13.15-15.00 in the seminar room 2112B of Automatic Control.