Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - January 10, 2017, 2 pm – 7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each problem.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises, old exams, and solutions of these, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

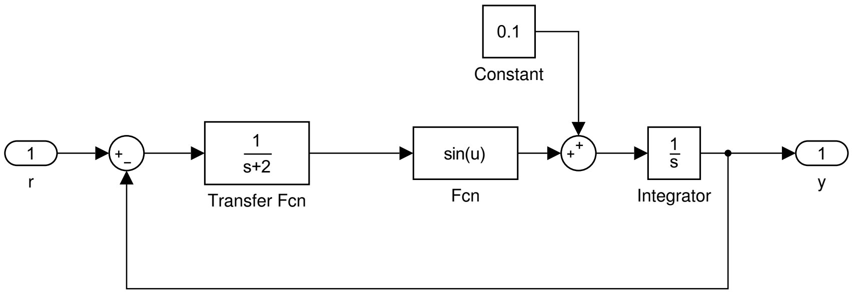

Figure 1 Block diagram of system with nonlinearity for Question 1.

1. Write the system in Figure 1 in the form:

$$\dot{x} = f(x) + Br$$
$$y = Cx$$

(2 p)

Solution

Let x_1 be the state after the integrator and x_2 be the state after the low pass filter. Then

$$\begin{aligned} x_1 &= \frac{1}{s}(0.1 + \sin(x_2)) \\ x_2 &= \frac{1}{s+2}(r - x_1) \\ &\Rightarrow \\ \dot{x}_1 &= 0.1 + \sin(x_2) \\ \dot{x}_2 &= r - x_1 - 2x_2 \end{aligned}$$

so $f(x) = \begin{bmatrix} 0.1 + \sin(x) \\ -x_1 - 2x_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \quad 0]$.

2. Determine and classify the equilibrium points (as e.g. stable node, unstable focus, etc.) of the system below.

$$\dot{x}_1 = \cos(x_1) - 1 + x_2$$
$$\dot{x}_2 = \sin(x_1)$$

(3 p)

Solution

First, we determine the equilibrium points, for which

$$0 = \cos(x_1) - 1 + x_2$$
$$0 = \sin(x_1)$$

The second equation yields that $x_1 = n\pi$, where n is any integer.
Let $f(x) = \begin{pmatrix} \cos(x_1) - 1 + x_2 \\ \sin(x_1) \end{pmatrix}$. Then, $\frac{\mathrm{d}f}{\mathrm{d}x} = \begin{pmatrix} -\sin(x_1) & 1 \\ \cos(x_1) & 0 \end{pmatrix}$.

For even values of n, $\cos(x_1) = 1$. For odd values, $\cos(x_1) = -1$. We consider these cases separately.

$\underline{n \text{ even}}$:

$0 = \cos(x_1) - 1 + x_2 = x_2$
Hence, the equilibrium points are $x = (n\pi, 0)$. Inserting any equilibrium point in $\frac{\mathrm{d}f}{\mathrm{d}x}$ gives the system matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_{1,2} = \pm 1$. Hence, we classify these points as saddle points.

$\underline{n \text{ odd}}$:

$0 = \cos(x_1) - 1 + x_2 = -2 + x_2 \Leftrightarrow x_2 = 2$
Hence, the equilibrium points are $x = (n\pi, 2)$. The system matrix is given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_{1,2} = \pm i$. Hence, we classify these points as center points.

3. The Nyquist diagram of a stable system $G(s)$ is shown in Figure 3. The system is in negative feedback with a nonlinearity $f_k(\cdot)$. Conclude if the closed system in Figure 2 is BIBO stable with each of the static nonlinearities f_1, f_2, f_3 and f_4. Use the Nyquist, Circle, Small Gain or Passivity theorem to show if the system is BIBO stable or not.

f_1, f_2 and f_3 are seen in Figure 4, f_4 is:

$$f_4(x) = \frac{(\text{sat}(x))^2}{4},$$

where $\text{sat}(\cdot)$ is a saturation between -1 and 1.

Hint: Linearization can sometimes be useful to show local properties. (4 p)

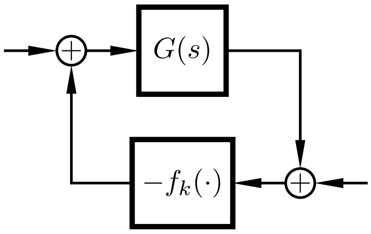

Figure 2 Feedback in Problem 3.

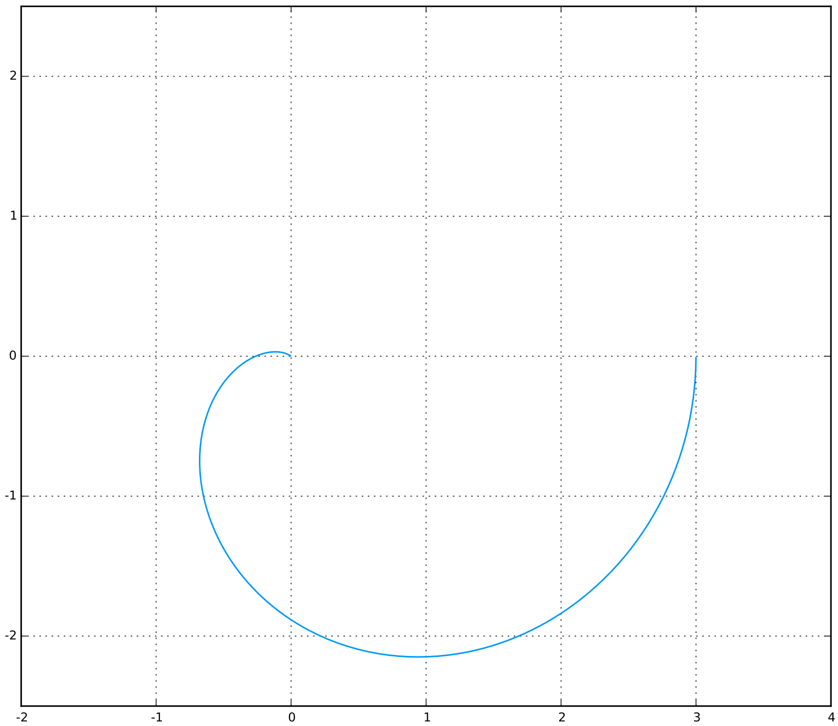

Figure 3 Nyquist diagram of system for Question 3.

Solution

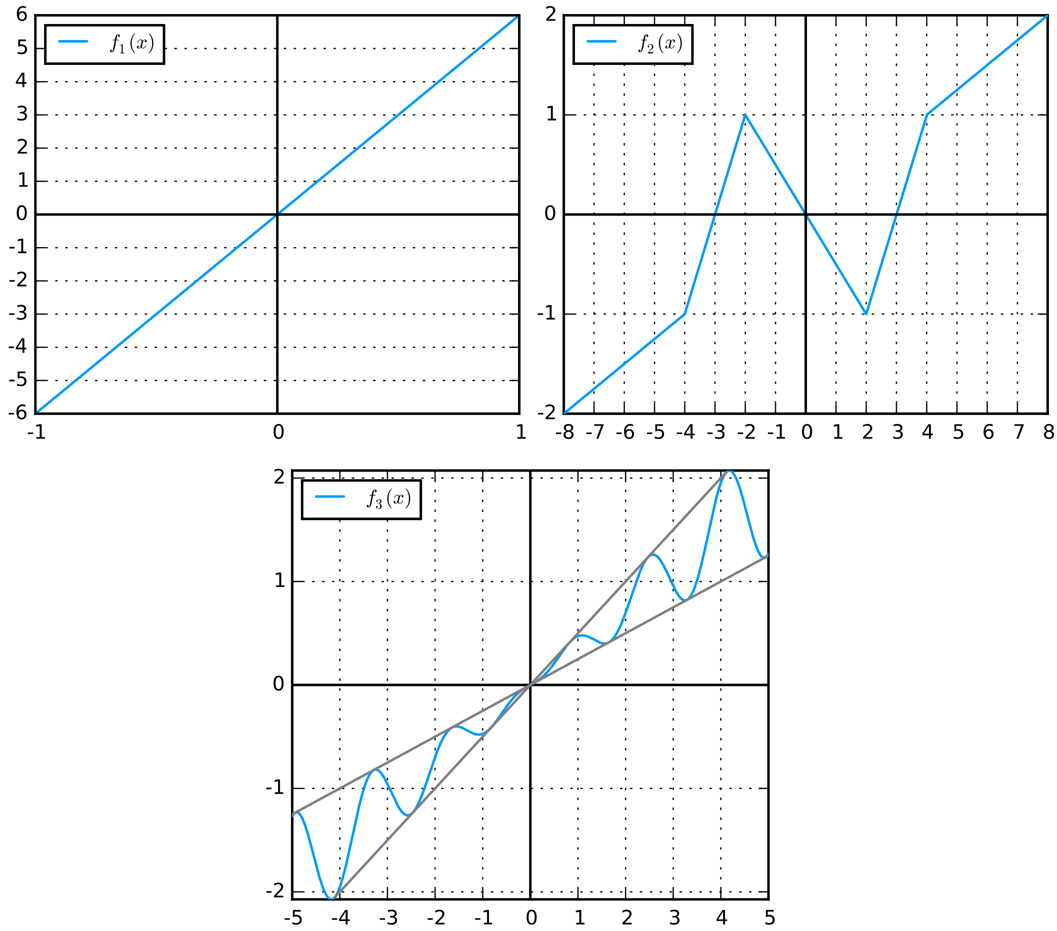

Figure 4 Functions from question 3. An upper and lower bound is shown for f_3.

a. f_1: Unstable. $f(x) = 6x$ is linear so the Nyquist Criterion says that the system is unstable if the curve encircles the point $-1/6$.

b. f_2: Not BIBO stable. The nonlinearity is contained by the slopes $k_1 = -\frac{1}{2}$ and $k_2 = \frac{1}{4}$. The Nyquist curve intersects the circle defined by $-\frac{1}{k_1} = 2$ and $-\frac{1}{k_2} = -4$ so the Circle Criterion is inconclusive. The Nyquist criterion can not be directly used since the function is not linear, and the Small Gain theorem is more restrictive than the Circle criterion for static nonlinearities. The nonleniarity does not satisfy the passivity theorem so we are not able to use this theorem either.
The nonlineariry is however locally linear with slope $-1/2$ around the origin. The linearization around origin therefore shows that the origin is unstable (the curve encircles the point 2). The maximum gain is therefore not finite and the system is not BIBO stable.

c. f_3: BIBO stable. The nonlinearity is contained by the slopes $k_1 = \frac{1}{4}$ and $k_2 = \frac{1}{2}$. The Nyquist does not intersect the circle defined by $-\frac{1}{k_1} = -4$ and $-\frac{1}{k_2} = -2$ so the Circle criterion guarantees stability.

d. f_4: BIBO stable. The gain is given by $\max_x \left\{ \left| \frac{f_5(x)}{x} \right| \right\}$. We have

$$\frac{f_5(x)}{x} = \frac{(\mathrm{sat}(x))^2}{4x} = \begin{cases} \dfrac{x}{4} & \text{if } |x| \le 1 \\ \dfrac{1}{4x} & \text{if } |x| \ge 1 \end{cases},$$

so $|\frac{f_5(x)}{x}| \le \frac{1}{4}$ and the gain is $\frac{1}{4}$. The largest amplitude of the Nyqust curve is 3 so the interconnection is stable by the small gain theorem.

4. Consider the system

$$\dot{x}_1 = x_2 + x_1(9 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(9 - x_1^2 - x_2^2)$$

Note that the subproblems can be solved independently of each other.

a. Denote by a a positive, non-zero, real, constant. For what a is the trajectory $(x_1, x_2) = (a\sin(t), a\cos(t))$ a solution to the system?

b. The trajectory in (a) is periodic and can thus be seen as a limit cycle. Determine whether this limit cycle is stable or not.

c. Linearize the system around the limit cycle.

(4 p)

Solution

a. Yes. Inserting the suggested solution into the system dynamics, and solving for a, yields $a = 3$.

b. To determine stability of the limit cycle, we introduce polar coordinates. With $r \geq 0$:

$$x_1 = r\cos(\theta)$$
$$x_2 = r\sin(\theta)$$

Differentiating both sides gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}$$

Inverting the matrix gives:

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Plugging in the state equations results in:

$$\dot{r} = r(a^2 - r^2) \tag{1}$$
$$\dot{\theta} = -1 \tag{2}$$

We see that the the only equilibrium points to (1) are 0 and a (since $r \geq 0$). Linearizing around $r = a$ (i.e. the limit cycle) gives:

$$\dot{\tilde{r}} = -2a^2\tilde{r}$$

which implies that $r = a$ is a locally asymptotically stable equilibrium point of (1). Hence the limit cycle is stable.

c. In compact notation we have:

$$\dot{x} = f(x)$$

Introduce $\tilde{x}(t) = x(t) - x_0(t)$ as the deviation from the nominal trajectory. We have

$$\dot{x} = \dot{x}_0 + \dot{\tilde{x}}$$

and the first order Taylor expansion of f around $x_0(t)$ is given by

$$\dot{x} = f(x_0) + \frac{\partial f(x_0)}{\partial x}\tilde{x}$$

So

$$\dot{x}_0 + \dot{\tilde{x}} = f(x_0) + \frac{\partial f(x_0)}{\partial x}\tilde{x}$$

Since $x_0(t)$ is a solution to the state equation we have $\dot{x}_0 = f(x_0)$ and thus

$$\dot{\tilde{x}} = \frac{\partial f(x_0(t))}{\partial x}\tilde{x} = A(t)\tilde{x}$$

where

$$A(t) = \begin{pmatrix} \frac{\partial f_1(x_0(t))}{\partial x_1} & \frac{\partial f_1(x_0(t))}{\partial x_2} \\ \frac{\partial f_2(x_0(t))}{\partial x_1} & \frac{\partial f_2(x_0(t))}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -2a^2\sin^2(t) & 1 - 2a^2\sin(t)\cos(t) \\ -1 - 2a^2\sin(t)\cos(t) & -2a^2\cos^2(t) \end{pmatrix}.$$

5. Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_2^2 + x_1 + u.\end{aligned}$$

a. Find a feedback law $u(x)$ so that the origin is a globally asymptotically stable equilibrium.

b. Is it possible to make the point $(0,1)$ a globally asymptotically stable equilibrium?

c. Find a feedback $u(x)$ so that the point $(1,0)$ is a globally asymptotically stable equilibrium. (3 p)

Solution

a. Let u cancel the nonlinearity, for example $u(x) = -x_2^2 - x_1 + r(x)$. The system can then be stabilized using linear control from $r(x)$. For example, by letting $r(x) = -x_1 - 2x_2$ we get

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - 2x_2\end{aligned}$$

which has the characteristic polinomial s^2+2s+1, with poles in -1. The system is linear so the stability is global. The control law is $u(x) = -x_2^2 - 2x_1 - 2x_2$.

b. No, all equilibrium points require $\dot{x}_1 = 0 = x_2$.

c. By doing a variable change $\tilde{x}_1 = x_1 - 1$, we get

$$\begin{aligned}\dot{\tilde{x}}_1 &= x_2\\ \dot{x}_2 &= x_2^2 + \tilde{x}_1 + 1 + u.\end{aligned}$$

First cancelling the nonlinearity and constant with $u(\tilde{x}_1, x_2) = -1 - x_2^2 - \tilde{x}_1 + r$ and again selecting $r(x) = -\tilde{x}_1 - 2x_2$ gives

$$\begin{aligned}\dot{\tilde{x}}_1 &= \ x_2\\ \dot{x}_2 &= -\tilde{x}_1 - 2x_2\end{aligned}$$

which is globally asymptotically stable to $\tilde{x}_1 = x_2 = 0$, i.e. $x_1 = 1$, $x_2 = 0$. The control law is therefore $u(x) = -1 - x_2^2 - 2\tilde{x}_1 - 2x_2 = 1 - x_2^2 - 2x_1 - 2x_2$.

6. Consider the function $f(x)$ in Figure 5.

a. Compute the describing function $N(A)$ for $f(x)$.
Hint: $\sin(x)^2 = \frac{1}{2}(1 - \cos(2x))$.

b. What happens to $N(A)$ when $A \to \infty$? How can this be interpreted?
If you were not able to solve the previous question: What do you expect the result to be? (4 p)

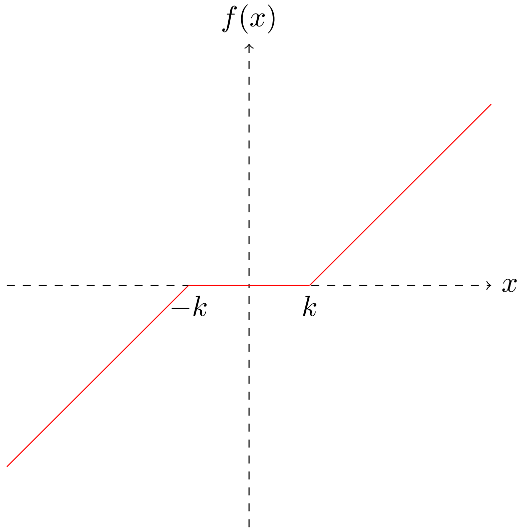

Figure 5 Nonlinearity in Problem 6. The slope is 1 outside the dead-zone.

Solution

a. The nonlinearity is static and odd so we only need to consider the output $f(A\sin(t))$ for $0 \leq t \leq \pi/2$. We have $f(x) = 0$ when $|x| < k$ so if $A < k$ then $f(A\sin(t)) \equiv 0$. Otherwise, when $x \geq k$ we have $f(x) = x - k$ so for $A \geq k$:

$$f(A\sin(t)) = \begin{cases} 0 & \text{if } t < \phi_f \\ A\sin(t) - k & \text{if } \phi_f \leq t \leq \pi/2 \end{cases}$$

where $\phi_f = \arcsin(k/A)$.

$$b_1 = \frac{4}{\pi}\int_{\phi_f}^{\pi/2} (A\sin(t) - k)\sin(t)\,dt = \frac{4}{\pi}\left(A\int_{\phi_f}^{\pi/2} \sin(t)^2\,dt - k\int_{\phi_f}^{\pi/2} \sin(t)\,dt\right) =$$

$$= \frac{4}{\pi}\left(\frac{A}{2}\int_{\phi_f}^{\pi/2} 1 - \cos(2t)\,dt + k\left[\cos(t)\right]_{\phi_f}^{\pi/2}\right) = \frac{4}{\pi}\left(\frac{A}{2}\left[t + \frac{1}{2}\sin(2t)\right]_{\phi_f}^{\pi/2} - k\cos(\phi_f)\right) =$$

$$= \frac{4}{\pi}\left(\frac{A}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin(\pi) - \phi_f + \frac{1}{2}\sin(2\phi_f)\right) - k\cos(\phi_f)\right)$$

Since $f(x)$ is odd we have $a_1 = 0$ and, for $A > k$:

$$N(A) = \frac{b_1}{A} = 1 - \frac{1}{\pi}\left(2\phi_f - \sin(2\phi_f)\right) - \frac{4k}{A\pi}\cos(\phi_f).$$

and $A \leq k$:

$$N(A) = 0.$$

The result can also be found by noting that the nonlinearity is the difference of a linear gain and a saturation (with $H = D = k$), and that $N_{f+g} = N_f + N_g$. This can result in the equivalent answer

$$N(A) = 1 - N_{\text{sat}}(A) = 1 - \frac{1}{\pi}\left(2\phi_f + \sin(2\phi_f)\right).$$

b. $\phi_f = \arcsin(\frac{k}{A}) \to 0$ so $\sin(2\phi_f) \to 0$ when $A \to \infty$. We therefore see that $N(A) \to 1$ when $A \to \infty$. This means that the output from the nonlinearity is neither suppressed nor amplified (at least the first harmonics) for large amplitudes. This is easily seen by that the function can be approximated by $f(x)/x \approx 1$ for large inputs.

7. The nonlinear system

$$\dot{x}_1 = x_2 - 2x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 - 2x_2(x_1^2 + x_2^2)$$

has the origin as an equilibrium point. Is the system globally asymptotically stable? (2 p)

Solution

Let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. We see directly that

- $V(0) = 0$
- $V(x) > 0, \forall x \neq 0$
- $V(x) \to \infty$ as $||x|| \to \infty$

Further, we have $\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2 - 2x_1(x_1^2 + x_2^2)) + x_2(-x_1 - 2x_2(x_1^2 + x_2^2)) = -2x_1^2(x_1^2 + x_2^2) - 2x_2^2(x_1^2 + x_2^2) = -2(x_1^2 + x_2^2)^2$
$< 0 \quad \forall x \neq 0$.
Therefore, we conclude that the origin is globally asymptotically stable.

8. Derive the optimal input trajectory for the following problem.

$$\text{minimize } \int_0^1 u(t)^2\, dt + x(1)^2$$

Subject to

$$\dot{x} = 2u$$
$$x(0) = 3$$

(3 p)

Solution

We have a fixed final time, $t_f = 1$. In the following, we use Theorem 18.2 in (Glad, T., and Ljung,L., "Reglerteori: Flervariabla och olinjära metoder"). We define
$\Phi(x) = x^2$
$L = u^2$
$f(x, u) = 2u$

Hamiltonian:

$$H = L + \lambda f = u^2 + 2\lambda u$$

Adjoint equation:

$$\dot{\lambda} = -H_x = 0$$
$$\lambda(t_f) = \Phi_x(x(t_f)) = 2x(t_f)$$

Hence, λ is constant, $\lambda(t) = 2x(t_f)$.
Optimality conditions:
Minimizing H with respect to u gives

$$H_u = 2u + 2\lambda = 0 \Leftrightarrow u = -\lambda$$

Hence, $u = -2x(t_f)$. Inserting this in the system equation yields

$$\dot{x} = -4x(t_f) \Leftrightarrow x(t_f) - x(0) = \int_0^{t_f} -4x(t_f)\, dt \Leftrightarrow x(1) - 3 = -4x(1) \Leftrightarrow x(1) = \frac{3}{5}$$

Hence, the optimal control law is $u = -2x(t_f) = -\frac{6}{5}$