



LUND
UNIVERSITY

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - January 13, 2016, 2 pm – 7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”/”Collection of Formulae”. Pocket calculator.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

1. Consider the control system

$$\ddot{z} = z - 2\text{sat}(\dot{z} + z) + u,$$

where

$$\text{sat}(y) = \begin{cases} 1 & \text{if } y > 1 \\ y & \text{if } -1 \leq y \leq 1 \\ -1 & \text{if } y < -1 \end{cases}$$

is the saturation function and u is a control variable.

- a. Rewrite the system in standard state-space form. (1 p)
- b. Is the origin a locally asymptotically stable equilibrium for the system above when the control $u = 0$? Motivate your answer. (1 p)
- c. Is the origin globally asymptotically stable when the control $u = 0$? Motivate your answer. (1 p)
- d. Design a feedback control $u(z, \dot{z})$ that makes the origin a globally asymptotically stable equilibrium. (1 p)

Solution

- a. Let $x_1 = z$ and $x_2 = \dot{z}$. Then,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - 2\text{sat}(x_1 + x_2) + u. \end{aligned}$$

- b. For (x_1, x_2) close to the origin we have that $\text{sat}(x_1 + x_2) = x_1 + x_2$. Thus the linearization of the system around the origin is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2. \end{aligned}$$

The characteristic polynomial of the associated matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ is $s^2 + 2s + 1$, so that A has a unique eigenvalue $\lambda = -1$. Since the linearization around the origin is stable, by Lyapunov's second theorem, the origin is locally asymptotically stable.

- c. The system has an equilibrium in $(x_1^*, x_2^*) = (2, 0)$. A trajectory starting in this point will never reach the origin.
- d. We can use exact linearization techniques by choosing the control in such a way that the nonlinearity is erased, and the closed-loop controlled system is linear and asymptotically stable. This is achieved, e.g., by

$$u(z, \dot{z}) = 2\text{sat}(\dot{z} + z) - 2(z + \dot{z}).$$

Indeed, in this case, the closed-loop controlled system $\ddot{z} = -z - 2\dot{z}$ has standard form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2, \end{aligned}$$

hence it is globally asymptotically stable.

2. Find and classify (stable/unstable node/focus, saddle point, center) all equilibrium points of the system

$$\begin{aligned}\dot{x}_1 &= \sin(2x_1 + x_2) \\ \dot{x}_2 &= x_1(x_2 - 1)\end{aligned}$$

that lie in the non-negative quadrant ($x_1 \geq 0, x_2 \geq 0$). (3 p)

Solution

At an equilibrium point $\dot{x}_1 = 0, \dot{x}_2 = 0$. For $\dot{x}_2 = 0$ to hold we must have that $x_1 = 0$ or $x_2 = 1$.

If $x_1 = 0$, we must have $x_2 = n_1\pi$, in order to have $\dot{x}_1 = 0$.

If $x_2 = 1$, we must have $x_1 = \frac{n_2\pi - 1}{2}$ in order to have $\dot{x}_1 = 0$.

Thus the equilibrium points in the non-negative quadrant are given by

$$(0, n_1\pi), \quad \left(\frac{n_2\pi - 1}{2}, 1\right), \quad \text{where } n_1, n_2 \text{ are integers and } n_1 \geq 0 \text{ and } n_2 \geq 1.$$

The system matrix for the linearization of the system around an equilibrium point (x_1^0, x_2^0) is given by

$$A(x_1^0, x_2^0) = \left. \frac{\partial f(x_1, x_2)}{\partial x} \right|_{(x_1, x_2) = (x_1^0, x_2^0)} = \begin{bmatrix} 2 \cos(2x_1^0 + x_2^0) & \cos(2x_1^0 + x_2^0) \\ (x_2^0 - 1) & x_1^0 \end{bmatrix}.$$

For $(x_1^0, x_2^0) = (0, n_1\pi)$ with n_1 even,

$$A(x_1^0, x_2^0) = \begin{bmatrix} 2 & 1 \\ n_1\pi - 1 & 0 \end{bmatrix}$$

which has the characteristic equation $s^2 - 2s - (n_1\pi - 1)$. This corresponds to a saddle point for $n_1 \geq 1$, and an unstable node for $n_1 = 0$.

For $(x_1^0, x_2^0) = (0, n_1\pi)$ with n_1 odd,

$$A(x_1^0, x_2^0) = \begin{bmatrix} -2 & -1 \\ n_1\pi - 1 & 0 \end{bmatrix}$$

which has the characteristic equation $s^2 + 2s + (n\pi - 1)$. This corresponds to an stable focus for all $n_1 \geq 0$.

For $(x_1^0, x_2^0) = ((n_2\pi - 1)/2, -1)$ with n_2 even,

$$A(x_1^0, x_2^0) = \begin{bmatrix} 2 & 1 \\ 0 & (n_2\pi - 1)/2 \end{bmatrix}$$

which has the characteristic equation $(s - 2)(s - (n_2\pi - 1)/2)$. This corresponds to an unstable node for all $n_2 \geq 1$.

For $(x_1^0, x_2^0) = ((n_2\pi - 1)/2, -1)$ with n_2 odd,

$$A(x_1^0, x_2^0) = \begin{bmatrix} -2 & -1 \\ 0 & (n_2\pi - 1)/2 \end{bmatrix}$$

which has the characteristic equation $(s + 2)(s - (n_2\pi - 1)/2)$. This corresponds to a saddle point for all $n_2 \geq 0$.

3. Consider the following system

$$\begin{aligned}\dot{x} &= y - x \\ \dot{y} &= x(1 - z) - y \\ \dot{z} &= xy - z.\end{aligned}$$

Use the candidate Lyapunov function

$$V(x, y, z) = \frac{x^2 + y^2 + z^2}{2}$$

to prove global asymptotic stability of the origin $(0, 0, 0)$. (2 p)

Solution

We have to check if V is Lyapunov function. Clearly, V is positive definite and radially unbounded. On the other hand,

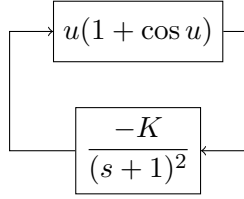
$$\begin{aligned}\dot{V} &= x\dot{x} + y\dot{y} + z\dot{z} \\ &= x(y - x) + y((1 - z) - y) + (xy - z)z \\ &= -x^2 - y^2 + 2xy \\ &= -(x - y)^2 - z^2 \\ &\leq 0,\end{aligned}$$

so that V has nonpositive drift. Now, observe that $\dot{V}(x, y, z) = 0$ on the whole set

$$E = \{(x, y, z) : x = y, z = 0\}.$$

Therefore, Lyapunov's theorems allow us to prove only local stability of the origin. For asymptotic stability we need to resort to La Salle's theorem. For that, we look for the largest invariant subset $M \subseteq E$. Observe that, for every $(x, y, z) \in E$, $\dot{z} = xy - x = x^2$. Hence $\dot{z} > 0$ for all $(x, y, z) \in E$ except the origin $(0, 0, 0)$. It follows that the largest invariant subset of E is $M = \{(0, 0, 0)\}$, so that LaSalle's theorem guarantees global asymptotic stability of the origin.

4. Consider the following system obtained as the closed-loop interconnection of a linear time invariant system with transfer function $G(s) = K/(s + 1)^2$, where $K > 0$ is a positive constant, with a static nonlinearity $f(u) = u(1 + \cos u)$



- a. Determine the values of K for which the small-gain theorem guarantees that the closed-loop interconnected system in the figure above is stable. (2 p)
- b. Determine the values of K for which the circle criterion guarantees that the closed-loop interconnected system in the figure above is stable. (2 p)

Solution

- a. Since $\max_u |1 + \cos u| = 2$, one gets that the gain of $f(u)$ is

$$\gamma_f = \sup_u \frac{|f(u)|}{|u|} = 2.$$

On the other hand, the gain of the LTI system with transfer function $G(s)$ is

$$\gamma_G = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \frac{K}{1 + \omega^2} = K.$$

The small-gain theorem guarantees stability of closed-loop interconnection system when $2K = \gamma_f \cdot \gamma_G < 1$, i.e., when $K < 1/2$.

- b. Observe that $0 \leq f(u) \leq 2u$ for $u \geq 0$, while $2u \leq f(u) \leq 0$ for $u \leq 0$. Hence, the nonlinearity belongs to the sector $[0, 2]$. Since the LTI system with transfer function $G(s) = \frac{K}{(s+1)^2}$ is stable for all values of K , the circle criterion guarantees stability of the closed loop interconnection if the Nyquist curve of $G(s)$ lies on the right-hand side of the line $\text{Re}(z) = -1/2$.

We thus want to find the values of K such that

$$\min_{\omega} \text{Re } G(i\omega) > -1/2.$$

This problem is relatively straight-forward to solve analytically,

$$\begin{aligned} \text{Re } G(i\omega) &= \text{Re} \frac{K}{(1 + i\omega)^2} \\ &= \text{Re} \frac{(1 - i\omega)^2}{(1 + \omega^2)^2} K \\ &= \text{Re} \frac{1 - \omega^2 - 2i\omega}{(1 + \omega^2)^2} K \\ &= \frac{1 - \omega^2}{(1 + \omega^2)^2} K =: h(\omega)K \end{aligned}$$

To find the minimum of $h(\omega)$ we find its stationary points as roots to $h'(\omega) = 0$,

$$\begin{aligned}h'(\omega) &= \frac{-2\omega}{(1+\omega^2)^2} + \frac{-4\omega(1-\omega^2)}{(1+\omega^2)^3} \\ &= \frac{-2\omega(1+\omega^2) - 4\omega(1-\omega^2)}{(1+\omega^2)^3} \\ &= -2\omega \frac{3-\omega^2}{(1+\omega^2)^3}\end{aligned}$$

and it is seen that $h(\omega)$ attains its minimal value for $\omega = \sqrt{3}$, with $h(\sqrt{3}) = -1/8$. Thus $\min_{\omega} \operatorname{Re} G(i\omega) = -K/8$, and $K < 4$ that guarantees that the closed loop is stable.

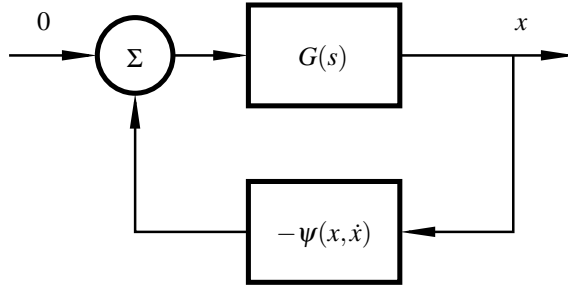


Figure 1 Negative feedback interconnection of Problem 5.

5. The so-called Van der Pol equation is given by

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0. \quad (1)$$

- a. Show that the system (1) can be written as the negative feedback interconnection of a linear time-invariant system $G(s)$ and a non-linearity $\psi(x, \dot{x})$ as indicated in Figure 1, where

$$G(s) = \frac{1}{s^2 - s + 1} \quad \psi(x, \dot{x}) = x^2 \dot{x} \quad (1 \text{ p})$$

- b. Using its definition, verify that the describing function of $\psi(x, \dot{x})$ is given by

$$N(A, \omega) = \frac{iA^2\omega}{4} \quad (1.5 \text{ p})$$

- c. Use the describing function method to compute the frequency and amplitude of all possible limit-cycles (with amplitude greater than 0) and determine whether they are locally stable or not. Hint: for determining stability consider the poles of the closed-loop system $G_{cl} = \frac{G_o}{1 + G_o}$, where $G_o(i\omega) := N(A, \omega)G(i\omega)$. (1.5 p)

Solution

- a. The system can be divided into the sum of a linear and non-linear part

$$(\ddot{x} - \dot{x} + x) + x^2 \dot{x} = 0 \Leftrightarrow (\ddot{x} - \dot{x} + x) = -\psi(x, \dot{x}) \quad (2)$$

According to block-diagram 1 it holds that $-\psi(x, \dot{x})$ is the input of $G(s)$. Hence, by denoting with $U(s)$ the input of $G(s)$ and Laplace-transforming (2) we get

$$X(s)(s^2 - s + 1) = U(s) \Leftrightarrow \frac{X(s)}{U(s)} = \frac{1}{s^2 - s + 1} = G(s). \quad (3)$$

b. $N(A, \omega) = \frac{b_1(\omega) + ia_1(\omega)}{A}$, where

$$a_1 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} A^3 \sin^2(\omega t) \cos^2(\omega t) \omega dt = \frac{\omega}{\pi} \int_0^{2\pi} A^3 \sin^2(\phi) \cos^2(\phi) d\phi = \frac{A^3 \omega}{4}$$

$$b_1 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} A^3 \sin^3(\omega t) \cos(\omega t) \omega dt = \frac{\omega}{\pi} \int_0^{2\pi} A^3 \sin^3(\phi) \cos(\phi) d\phi = 0$$

and therefore $N(A, \omega) = \frac{iA^2\omega}{4}$.

c. In order to have a limit-cycle it has to be fulfilled that

$$N(A, \omega)G(i\omega) = -1 \Leftrightarrow \frac{iA^2\omega}{4} = -\omega^2 - i\omega + 1$$

Hence, $\omega = 1$ and $A = 2$. For determining the stability we look at the poles the closed loop system $G_{cl} = \frac{G_o}{1 + G_o}$, where $G_o(i\omega) := N(A, \omega)G(i\omega)$, i.e.

$$G_o(s) = \frac{A^2 s}{4(s^2 - s + 1)} =: \frac{P(s)}{Q(s)}$$

and

$$G_{cl}(s) = \frac{P(s)}{Q(s) + P(s)} = \frac{A^2 s}{4(s^2 - s + 1) + A^2 s}$$

Hence, the characteristic equation is given by

$$s^2 - \left(1 - \frac{A^2}{4}\right) s + 1 = 0$$

and therefore

$$s_{1/2} = \frac{1}{2} \left(1 - \frac{A^2}{4}\right) \pm \sqrt{-1 + \frac{1}{4} \left(1 - \frac{A^2}{4}\right)^2}$$

Thus, $G_{cl}(s)$ is asymptotically stable if $A > 2$ and unstable if $A < 2$ and therefore the limit-cycle is stable.

There is another "limit-cycle" for the equilibrium point $x = 0$, however the amplitude is then 0.

6. Consider the sliding mode control system

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 2x_2 \\ \dot{x}_2 &= 2x_1 + x_2 - 2u \end{aligned} \quad u = \begin{cases} +1 & \text{if } x_1 + x_2 > 0 \\ -1 & \text{if } x_1 + x_2 < 0 \end{cases}.$$

a. Determine the sliding set. (1.5 p)

b. Find the sliding dynamics and determine whether the origin is a stable equilibrium for it. Motivate your answer. (1.5 p)

Solution

a. Let $\sigma(x_1, x_2) = x_1 + x_2$ and

$$f^+(x_1, x_2) = \begin{bmatrix} 3x_1 + 2x_2 \\ 2x_1 + x_2 - 2 \end{bmatrix}, \quad f^-(x_1, x_2) = \begin{bmatrix} 3x_1 + 2x_2 \\ 2x_1 + x_2 + 2 \end{bmatrix}.$$

Then, the sliding surface is $\Sigma = \{(x_1, x_2) : \sigma(x) = 0\} = \{(x_1, x_2) : x_1 = -x_2\}$, and the sliding set is the subset of Σ where

$$f^+(x_1, x_2) \cdot \nabla\sigma(x_1, x_2) < 0, \quad f^-(x_1, x_2) \cdot \nabla\sigma(x_1, x_2) > 0.$$

On the surface Σ the inequality above reduce to

$$3x_1 + 2x_2 + 2x_1 + x_2 - 2 = 2x_1 - 2 < 0, \quad 3x_1 + 2x_2 + 2x_1 + x_2 + 2 = 2x_1 + 2 > 0,$$

so that the sliding set is

$$\{(x_1, x_2) : x_1 = -x_2, -1 < x_1 < 1\}.$$

b. The sliding dynamics on the sliding set is obtained as

$$\dot{x} = \alpha f^+(x) + (1 - \alpha)f^-(x),$$

where α is such that

$$(\alpha f^+(x) + (1 - \alpha)f^-(x)) \cdot \nabla\sigma(x_1, x_2) = 0.$$

On the sliding set, the equation above reduces to $2x_1 - 2\alpha + 2(1 - \alpha) = 0$, i.e.,

$$\alpha = (1 + x_1)/2.$$

By substituting this value of α we obtain the sliding dynamics

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2, \end{aligned}$$

which is unstable.

7. Consider the following optimal flow control problem. There are two cells containing quantities x_1 kg and x_2 kg, respectively, of the same incompressible fluid. Cell 1 gets a constant positive external inflow of rate $v = 1$ kg/s. The two cells are connected in such a way that the fluid can flow from cell 1 to cell 2 at a controlled flow rate of u kg/s that satisfies the constraints $0 \leq u \leq C$ where $C > 0$ is the maximum flow capacity. Cell 2 has an outflow towards the external world of rate $w = x_2$. The law of mass conservation then gives the following dynamical equations:

$$\dot{x}_1 = 1 - u, \quad \dot{x}_2 = u - x_2.$$

Given that the initial quantities of fluid in the two cells, $x_1(0) \geq 0$ and $x_2(0) \geq 0$, are known, we are interested in solving the following open-loop optimal control problem:

$$\min_{u(t): 0 \leq t \leq 1} \int_0^1 (\alpha x_1(t) + x_2(t)) dt,$$

where $\alpha > 0$ is a parameter.

- a. Find the Hamiltonian of the problem. (1 p)
- b. Determine the adjoint equations and the final time conditions on the co-state variables. (1 p)
- c. Solve the adjoint equations. (1 p)
- d. Use the Pontryagin maximum principle to determine the optimal control $u^*(t)$ when $\alpha = 1$, $\alpha = 3/4$, and $\alpha = 1/4$. (Bonus: provide a physical justification of the different solutions obtained in the three different cases.) (2 p)

Solution

This is a classical optimal control problem with two-dimensional dynamics

$$\dot{x}_1 = f_1(x_1, x_2, u) = 1 - u, \quad \dot{x}_2 = f_2(x_1, x_2, u) = u - x_2,$$

running cost $L(x_1, x_2, u) = \alpha x_1 + x_2$, no final state cost or constraints, control space $U = [0, C]$ and given final time $t_f = 1$. Since the final time is given and there no constraints on the final state, we can use the first formulation of the Pontryagin maximum principle.

- a. The Hamiltonian is given by

$$\begin{aligned} H(x_1, x_2, u, \lambda_1, \lambda_2) &= L(x_1, x_2, u) + \lambda_1 f_1(x_1, x_2, u) + \lambda_2 f_2(x_1, x_2, u) \\ &= \alpha x_1 + x_2 + \lambda_1(1 - u) + \lambda_2(u - x_2) \\ &= \alpha x_1 + x_2 + \lambda_1 - \lambda_2 x_2 + (\lambda_2 - \lambda_1)u. \end{aligned}$$

- b. The adjoint equations are given by

$$\dot{\lambda}_1 = \frac{\partial H}{\partial x_1} = -\alpha, \quad \dot{\lambda}_2 = \frac{\partial H}{\partial x_2} = \lambda_2 - 1.$$

Since there are no final costs, the final time conditions on the co-states simply read

$$\lambda_1(1) = 0, \quad \lambda_2(1) = 0.$$

- c. The solution of the adjoint equations above gives the following evolution for the co-states

$$\lambda_1(t) = \alpha(1 - t), \quad \lambda_2(t) = 1 - e^{t-1}.$$

- d. The Pontryagin maximum principle implies that the optimal control satisfies

$$u^*(t) \in \arg \min_{u \in U} \{H(x_1^*(t), x_2^*(t), u, \lambda_1(t), \lambda_2(t))\} = \arg \min_{u \in [0, C]} \{(\lambda_2(t) - \lambda_1(t))u\}.$$

Hence,

$$u^*(t) = \begin{cases} C & \text{if } \lambda_1(t) > \lambda_2(t) \\ 0 & \text{if } \lambda_1(t) < \lambda_2(t) \end{cases}$$

For $\alpha = 1$, one has that

$$\lambda_2(t) = 1 - e^{t-1} < 1 - t = \lambda_1(t), \quad \forall t \in [0, 1],$$

so that

$$u^*(t) = C, \quad \forall t \in [0, 1]$$

is the optimal control. For $\alpha = 1/4$, one has that

$$\lambda_2(t) = 1 - e^{t-1} > \frac{1}{4}(1 - t) = \lambda_1(t), \quad \forall t \in [0, 1],$$

so that

$$u^*(t) = 0, \quad \forall t \in [0, 1]$$

is the optimal control. Finally, for $\alpha = 3/4$, there exists $0 < t^* < 1$ such that

$$\lambda_2(t) = 1 - e^{t-1} < \frac{3}{4}(1 - t) = \lambda_1(t), \quad \forall t \in [0, t^*),$$

$$\lambda_2(t) = 1 - e^{t-1} > \frac{3}{4}(1 - t) = \lambda_1(t), \quad \forall t \in (t^*, 1],$$

so that the optimal control satisfies

$$u^*(t) = \begin{cases} C & \text{if } 0 \leq t < t^* \\ 0 & \text{if } t^* < t \leq 1. \end{cases}$$

To get a physical interpretation, observe that the law of mass conservation gives $\dot{x}_1 + \dot{x}_2 = 1 - x_2$, i.e., the growth rate of the total quantity of fluid in the two cells is equal to the external inflow $v = 1$ minus the external outflow $w = x_2$. When $\alpha = 1$, the running cost $L(x_1, x_2, u) = x_1 + x_2$ has derivative $\dot{L} = 1 - x_2$. Hence, in this case, there is an incentive in pushing as much fluid as possible to the second cell, so that the growth rate of the running cost is as little as possible. For values of α smaller than 1, there is a tradeoff between the benefit of moving fluid from the first cell to the second one, so that there is a larger outflow from the cells towards the external world, and the higher cost x_2 of keeping the fluid in cell 2 as opposed to the cost αx_1 keeping it cell 1.