Real-Time Systems

Formula Sheet

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Aliasing

The fundamental alias for a frequency ω_1 is given by

$$\omega = |(\omega_1 + \omega_N) \text{ mod } (\omega_s) - \omega_N|$$

ZOH-Sampling of a system with input time delay $\tau \leq h$

ZOH-sampling the continuous-time system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t - \tau)$$

gives

$$x(kh + h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh - h)$$

where

$$\begin{aligned} \Phi &= e^{Ah} \\ \Gamma_0 &= \int_0^{h-\tau} e^{As}\, ds\; B \\ \Gamma_1 &= e^{A(h-\tau)} \int_0^{\tau} e^{As}\, ds\; B \end{aligned}$$

Different ways of calculating Φ

$$\begin{aligned} \Phi &= e^{Ah} = I + Ah + A^2h^2/2 + \ldots \\ \Phi &= \mathcal{L}^{-1}(sI - A)^{-1} \end{aligned}$$

Solution to system equation

The solution to the system equation

$$\begin{aligned} x(kh + h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh) + Du(kh) \end{aligned}$$

is given by

$$Y(z) = C(zI - \Phi)^{-1} z x(0) + \Big(C(zI - \Phi)^{-1}\Gamma + D\Big) U(z)$$

Mapping of second-order poles

When a continuous-time system with the characteristic polynomial

$$s^2 + 2\zeta\omega_0 s + \omega_0^2$$

is ZOH-sampled the corresponding discrete-time characteristic polynomial is given by

$$z^2 + a_1 z + a_2$$

with

$$\begin{aligned} a_1 &= -2e^{-\zeta\omega_0 h} \cos\left(\sqrt{1 - \zeta^2}\,\omega_0 h\right) \\ a_2 &= e^{-2\zeta\omega_0 h} \end{aligned}$$

Rules of thumb for sampling interval selection

Approximation of continuous-time design:

$$h\omega_c \approx 0.05 \text{ to } 0.15$$

where ω_c is the cross-over frequency (in radians per second) of the continuous-time system.

Discrete-time design:

$$\omega h = 0.1 \text{ to } 0.6$$

where ω is the desired natural frequency of the closed-loop system.

Some discrete-time functions and corresponding z-transforms

$f(k)$		$\mathcal{Z}(f(k))$
$\delta(k)$	(pulse)	1
1	(step)	$\frac{z}{z-1}$
k	(ramp)	$\frac{z}{(z-1)^2}$
a^k		$\frac{z}{z-a}$

Some properties of the z-transform

1. Definition
   $F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k}$
2. Inversion
   $f(kh) = \frac{1}{2\pi i} \oint F(z) z^{k-1}\, dz$
3. Linearity
   $\mathcal{Z}\{\alpha f + \beta g\} = \alpha \mathcal{Z} f + \beta \mathcal{Z} g$
4. Time shift
   $\mathcal{Z}\{q^{-n} f\} = z^{-n} F$
   $\mathcal{Z}\{q^{n} f\} = z^{n}(F - F_1)$ where $F_1(z) = \sum_{j=0}^{n-1} f(jh) z^{-j}$
5. Initial-value theorem
   $f(0) = \lim_{z\to\infty} F(z)$
6. Final-value theorem
   If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle, then $\lim_{k\to\infty} f(kh) = \lim_{z\to 1}(1 - z^{-1})F(z)$.
7. Convolution
   $\mathcal{Z}\{f * g\} = \mathcal{Z}\left\{\sum_{n=0}^{k} f(n) g(k-n)\right\} = (\mathcal{Z} f)(\mathcal{Z} g)$

Zero-order hold sampling of a continuous-time system with transfer function $G(s)$

$G(s)$	$H(z) = \dfrac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$	
$\dfrac{1}{s}$	$\dfrac{h}{z-1}$	
$\dfrac{1}{s^2}$	$\dfrac{h^2(z+1)}{2(z-1)^2}$	
e^{-sh}	z^{-1}	
$\dfrac{a}{s+a}$	$\dfrac{1-\exp(-ah)}{z-\exp(-ah)}$	
$\dfrac{a}{s(s+a)}$	$b_1 = \dfrac{1}{a}\left(ah - 1 + e^{-ah}\right)$ $a_1 = -(1+e^{-ah})$	$b_2 = \dfrac{1}{a}\left(1 - e^{-ah} - ahe^{-ah}\right)$ $a_2 = e^{-ah}$
$\dfrac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1+ah)$ $a_1 = -2e^{-ah}$	$b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_2 = e^{-2ah}$
$\dfrac{s}{(s+a)^2}$	$\dfrac{(z-1)he^{-ah}}{(z-e^{-ah})^2}$	
$\dfrac{ab}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \dfrac{b(1-e^{-ah}) - a(1-e^{-bh})}{b-a}$ $b_2 = \dfrac{a(1-e^{-bh})e^{-ah} - b(1-e^{-ah})e^{-bh}}{b-a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$	
$\dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha\left(\beta + \dfrac{\zeta\omega_0}{\omega}\gamma\right)$ $b_2 = \alpha^2 + \alpha\left(\dfrac{\zeta\omega_0}{\omega}\gamma - \beta\right)$ $a_1 = -2\alpha\beta$ $a_2 = \alpha^2$	$\omega = \omega_0\sqrt{1-\zeta^2} \qquad \zeta < 1$ $\alpha = e^{-\zeta\omega_0 h}$ $\beta = \cos(\omega h)$ $\gamma = \sin(\omega h)$

Observers

Observer (predictor case):

$$\hat{x}(k+1 \mid k) = \Phi\hat{x}(k \mid k-1) + \Gamma u(k) + K\Big(y(k) - C\hat{x}(k \mid k-1)\Big)$$

Observer (filter case, with direct term):

$$\begin{aligned}\hat{x}(k \mid k) &= \Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1) \\ &\quad + K\Big[y(k) - C\Big(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\Big)\Big] \\ &= (I - KC)\Big(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\Big) + Ky(k)\end{aligned}$$

Approximation of continuous-time designs

Forward difference:

$$s' = \frac{z-1}{h}$$

Backward difference:

$$s' = \frac{z-1}{zh}$$

Tustin:

$$s' = \frac{2}{h} \cdot \frac{z-1}{z+1}$$

Tustin with prewarping:

$$s' = \frac{\omega_1}{\tan(\omega_1 h/2)} \cdot \frac{z-1}{z+1}$$

Approximation of continuous-time state feedback

$$\begin{aligned}u(t) &= Mu_c(t) - Lx(t) \\ u(kh) &= \tilde{M}u_c(kh) - \tilde{L}x(kh) \\ \tilde{L} &= L\Big(I + (A - BL)h/2\Big) \\ \tilde{M} &= (I - LBh/2)M\end{aligned}$$

EDF schedulability condition

$$U = \sum_{i=1}^{n} \frac{C_i}{T_i} \leq 1$$

RM schedulability conditions

$$\sum_{i=1}^{n} \frac{C_i}{T_i} \leq n(2^{1/n} - 1)$$

$$\prod_{i=1}^{i=n} (\frac{C_i}{T_i} + 1) \leq 2$$

DM schedulability condition

$$\sum_{i=1}^{n} \frac{C_i}{D_i} \leq n(2^{1/n} - 1)$$

Response time calculation

$$R_i = C_i + \sum_{\forall j \in hp(i)} \left\lceil \frac{R_i}{T_j} \right\rceil C_j$$

The Laplace transform

Operator lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Linearity
2	$F(s+a)$	$e^{-at} f(t)$	Damping
3	$e^{-as} F(s)$	$\begin{cases} f(t-a) & t-a>0 \\ 0 & t-a<0 \end{cases}$	Time delay
4	$\frac{1}{a} F\left(\frac{s}{a}\right) \quad (a>0)$	$f(at)$	Scaling in t-domain
5	$F(as) \quad (a>0)$	$\frac{1}{a} f\left(\frac{t}{a}\right)$	Scaling in s-domain
6	$F_1(s)F_2(s)$	$\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$	Convolution in t-domain
7	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\sigma) F_2(s-\sigma)\, d\sigma$	$f_1(t) f_2(t)$	Convolution in s-domain
8	$sF(s) - f(0)$	$f'(t)$	Differentiation in t-domain
9	$s^2 F(s) - s f(0) - f'(0)$	$f''(t)$	
10	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	$f^{(n)}(t)$	
11	$\frac{d^n F(s)}{ds^n}$	$(-t)^n f(t)$	Differentiation in s-domain
12	$\frac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$	Integration in t-domain
13	$\int_s^\infty F(\sigma)\, d\sigma$	$\frac{f(t)}{t}$	Integration in s-domain
14	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final value theorem
15	$\lim_{s\to\infty} sF(s)$	$\lim_{t\to 0} f(t)$	Initial value theorem

Transform lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	1	$\delta(t)$	Dirac function
2	$\frac{1}{s}$	1	Step function
3	$\frac{1}{s^2}$	t	Ramp function
4	$\frac{1}{s^3}$	$\frac{1}{2}t^2$	Acceleration
5	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	
6	$\frac{1}{s+a}$	e^{-at}	
7	$\frac{1}{(s+a)^2}$	$t \cdot e^{-at}$	
8	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
9	$\frac{1}{1+as}$	$\frac{1}{a}e^{-t/a}$	
10	$\frac{a}{s^2+a^2}$	$\sin at$	
11	$\frac{a}{s^2-a^2}$	$\sinh at$	
12	$\frac{s}{s^2+a^2}$	$\cos at$	
13	$\frac{s}{s^2-a^2}$	$\cosh at$	
14	$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$	
15	$\frac{1}{s(1+as)}$	$1-e^{-t/a}$	
16	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-bt}-e^{-at}}{a-b}$	

Transform lexicon, continued

	Laplace transform $F(s)$		Time function $f(t)$
17	$\dfrac{s}{(s+a)(s+b)}$		$\dfrac{ae^{-at}-be^{-bt}}{a-b}$
18	$\dfrac{a}{(s+b)^2+a^2}$		$e^{-bt}\sin at$
19	$\dfrac{s+b}{(s+b)^2+a^2}$		$e^{-bt}\cos at$
20	$\dfrac{1}{s^2+2\zeta\omega_0 s+\omega_0^2}$		
		$\zeta=0$	$\frac{1}{\omega_0}\sin\omega_0 t$
		$\zeta<1$	$\frac{1}{\omega_0\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)$
		$\zeta=1$	$te^{-\omega_0 t}$
		$\zeta>1$	$\frac{1}{\omega_0\sqrt{\zeta^2-1}}\,e^{-\zeta\omega_0 t}\sinh\left(\omega_0\sqrt{\zeta^2-1}\,t\right)$
21	$\dfrac{s}{s^2+2\zeta\omega_0 s+\omega_0^2}$		
	$0\le\tau\le\pi:$	$\zeta<1$	$\frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t+\tau\right)$
			$\tau=\arctan\frac{\omega_0\sqrt{1-\zeta^2}}{-\zeta\omega_0}$
		$\zeta=0$	$\cos\omega_0 t$
		$\zeta=1$	$(1-\omega_0 t)e^{-\omega_0 t}$
22	$\dfrac{a}{(s^2+a^2)(s+b)}$		$\dfrac{1}{\sqrt{a^2+b^2}}\left(\sin(at-\phi)+e^{-bt}\sin\phi\right)$
			$\phi=\arctan\dfrac{a}{b}$

Laplace transform table, continued

	Laplace transform $F(s)$	Time function $f(t)$
23	$\dfrac{s}{(s^2+a^2)(s+b)}$	$\dfrac{1}{\sqrt{a^2+b^2}}\left(\cos(at-\phi)-e^{-bt}\cos\phi\right)$ $\phi = \arctan\dfrac{a}{b}$
24	$\dfrac{ab}{s(s+a)(s+b)}$	$1+\dfrac{ae^{-bt}-be^{-at}}{b-a}$
25	$\dfrac{a^2}{s(s+a)^2}$	$1-(1+at)e^{-at}$
26	$\dfrac{a}{s^2(s+a)}$	$t-\frac{1}{a}(1-e^{-at})$
27	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(b-a)(c-a)(b-c)}$
28	$\dfrac{\omega_0^2}{s(s^2+2\zeta\omega_0 s+\omega_0^2)}$	
	$0<\zeta<1$	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t+\phi\right)$ $\phi = \arccos\zeta$
	$\zeta = 0$	$1-\cos\omega_0 t$
29	$\dfrac{1}{(s+a)^{n+1}}$	$\dfrac{1}{n!}t^n e^{-at}$
30	$\dfrac{s}{(s+a)(s+b)(s+c)}$	$\dfrac{a(b-c)e^{-at}+b(c-a)e^{-bt}+c(a-b)e^{-ct}}{(b-a)(b-c)(a-c)}$
31	$\dfrac{as}{(s^2+a^2)^2}$	$\dfrac{t}{2}\sin at$
32	$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
33	$\dfrac{1}{\sqrt{s}}F(\sqrt{s})$	$\dfrac{1}{\sqrt{\pi t}}\displaystyle\int_0^\infty e^{-\sigma^2/4t}f(\sigma)\,d\sigma$