The Ball and Beam Process

1. The Process

The ball and beam process consists of a horizontal beam and a motor. A rolling ball should be balanced on the beam. The beam angle is controlled by the motor.

The output signals from the process are the beam angle relative the horizontal plane (ϕ) and the position of the ball (x).

Process Model

The process model can be decomposed into the model of the angle process, $G_\phi(s)$, and a model of how the ball position is influenced by the beam angle, $G_x(s)$. The total transfer function from the input voltage to the voltage that indicates the ball position is then $G_\phi(s)G_x(s)$.

The angle process Neglecting the dynamics in the motor, the angular velocity of the beam is proportional to the motor voltage, i.e.

$$\dot{\phi}(t) = k_\phi u(t)$$

Hence

$$G_\phi(s) = \frac{k_\phi}{s}$$

Through experiments it has been determined that $k_\phi \approx 4.4$.

The ball position process The dynamics of the ball position can be determined by classical mechanics. Assume that the ball position is x, the beam angle is ϕ and that the beam lies in the center of rotation ϕ. Coordinates and forces are defined according to Figure 1. The force equations are

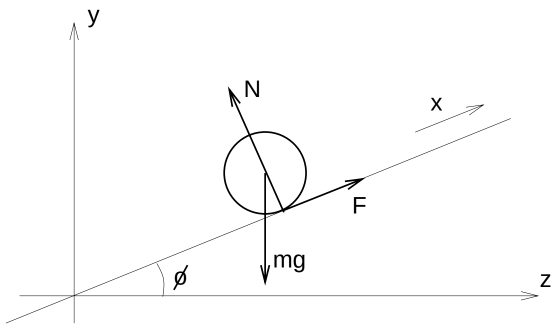

Figure 1 Forces acting on the ball on the beam.

$$m\ddot{y} = -mg + N\cos\phi + F\sin\phi$$
$$m\ddot{z} = -N\sin\phi + F\cos\phi$$

Multiply with $\sin\phi$ and $\cos\phi$ and sum:

$$m(\ddot{y}\sin\phi + \ddot{z}\cos\phi) = -mg\sin\phi + F \tag{1}$$

The condition for a rolling ball without friction is $Fr = J\dot{\omega} = -J\ddot{x}/r$ For a solid ball we have $J = 2mr^2/5$, hence $F = -2m\ddot{x}/5$. Further, we have $z = x\cos\phi$ and $y = x\sin\phi$. Derive these expressions twice, multiply with $\cos\phi$ and $\sin\phi$, sum. This gives

$$\ddot{y}\sin\phi + \ddot{z}\cos\phi = \ddot{x} - x\dot{\phi}^2$$

Insert this expression and the expression for F in (1). We then get

$$m(\ddot{x} - x\dot{\phi}^2) = -mg\sin\phi - \frac{2}{5}m\ddot{x}.$$

If we finally assume that ϕ and $\dot{\phi}$ are small we obtain

$$G_x(s) = -\frac{5g}{7s^2} = -\frac{k_x}{s^2}$$

where $k_x \approx 7.0$. Notice that the gain of this process is negative.

2. Optimal Reference Generation

To get fast and accurate setpoint response, it is necessary to introduce feedforward in the controller. This is known as the servo problem. Using optimal control theory, a time-optimal feedforward control signal, $u_{ff}(t)$, and the corresponding state trajectories, $\xi_r(t)$, can be derived. The optimal state trajectories can be used as reference values in the feedback controller.

Optimal Control Signal and State Trajectories

Since the process model is linear, the time-optimal control signal is of "bang-bang" character. Let the magnitude of the maximum control authority be denoted $\bar{u}$ (this value must be tuned). Suppose that the ball is at rest at time $t = 0$ and that it should be moved from the initial position x_0 to the final position x_f. The optimal control signal can be shown to be

$$u_{ff}(t) = \begin{cases} -\operatorname{sgn}(x_f - x_0)\bar{u}, & 0 \le t < T, \\ \operatorname{sgn}(x_f - x_0)\bar{u}, & T \le t < 3T, \\ -\operatorname{sgn}(x_f - x_0)\bar{u}, & 3T \le t \le 4T, \end{cases}$$

i.e., the ball reaches the final position at time $t = 4T$. Integrating the system equations, we get the following expressions for $\phi_r(t)$, $\dot{x}_r(t)$, and $x_r(t)$:

$$\phi_r(t) = \begin{cases} -\operatorname{sgn}(x_f - x_0)k_\phi\bar{u}\,t, & 0 \le t < T, \\ \operatorname{sgn}(x_f - x_0)k_\phi\bar{u}(t - 2T), & T \le t < 3T, \\ -\operatorname{sgn}(x_f - x_0)k_\phi\bar{u}(t - 4T), & 3T \le t \le 4T, \end{cases}$$

$$\dot{x}_r(t) = \begin{cases} \operatorname{sgn}(x_f - x_0)k_\phi k_x\bar{u}\,t^2/2, & 0 \le t < T, \\ -\operatorname{sgn}(x_f - x_0)k_\phi k_x\bar{u}(t^2/2 - 2Tt + T^2), & T \le t < 3T, \\ \operatorname{sgn}(x_f - x_0)k_\phi k_x\bar{u}(t^2/2 - 4Tt + 8T^2), & 3T \le t \le 4T, \end{cases}$$

$$x_r(t) = \begin{cases} x_0 + \operatorname{sgn}(x_f - x_0)k_\phi k_x\bar{u}\,t^3/6, & 0 \le t < T, \\ x_0 - \operatorname{sgn}(x_f - x_0)k_\phi k_x\bar{u}(t^3/6 - Tt^2 + T^2t - T^3/3), & T \le t < 3T, \\ x_0 + \operatorname{sgn}(x_f - x_0)k_\phi k_x\bar{u}(t^3/6 - 2Tt^2 + 8T^2t - 26T^3/3), & 3T \le t \le 4T. \end{cases}$$

Finally, setting $x_r(4T) = x_f$, we can solve for T:

$$T = \sqrt[3]{\frac{|x_f - x_0|}{2k_\phi k_x \bar{u}}}.$$

An example of an optimal control signal and the corresponding state trajectories is shown in Figure 2.

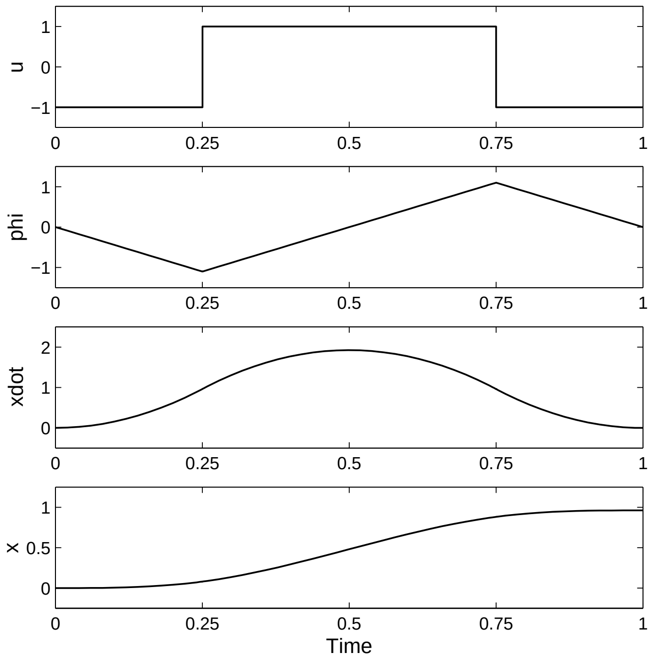

Figure 2 An example of an optimal control signal and the corresponding state trajectories.

Cascaded PID control When cascaded PID control is used, u_{ff} is added to the inner control signal, ϕ_r is added to the inner reference value, and x_r is used as the outer reference value. The remaining reference state, $\dot{x}_r$, is not used.