

Systems Engineering/Process Control

Hand-In Assignment 1

Analysis and Simulation of Concentration Dynamics

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1. Introduction

The purpose of the assignment is to study a class of systems known as compartment models, which play an important role in physics, chemistry, biology, pharmacokinetics, etc. You will be exposed to basic concepts and principles of automatic control and systems engineering.

You will use MATLAB with Control System Toolbox in the assignment.

The assignment is performed in groups of three students each and should take approximately one day to complete.

Background

You need to be familiar with the following concepts:

- state-space model
- stability
- transfer function
- poles and zeros
- open and closed system

2. Problem Formulation

We will study the properties of a compartment system with three compartments, see Figure 1. The model can for instance represent how a headache pill is absorbed by the body. Compartments 1, 2, and 3 could represent the stomach, the blood system, and the brain, while the outlet from compartment 2 could represent the secretion of the drug from the blood to the kidneys and the liver.

Historical note: One of the very first compartment models was actually developed at Lund University by Professor Erik Widmark, who studied the uptake of alcohol in the body.

Problem 1. *Compartment models can be used to model many different things. Mention some other applications that you have been exposed to during your study program that might be described by compartment models.* □

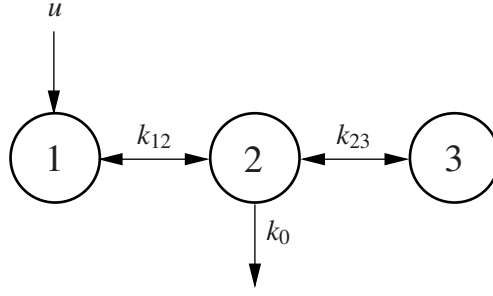


Figure 1 A compartment system with three compartments. The input signal u models the injection of a substance into compartment 1. The parameters k_{12} and k_{23} determine the exchange between adjacent compartments, while the parameter k_0 determines the elimination from compartment 2.

Mathematical Model

Let V_i be the volume of compartment i , and let c_i represent the concentration of the substance under study in compartment i . The equations for the system in Figure 1 can then be written as

$$\begin{aligned}
 V_1 \frac{dc_1}{dt} &= k_{12}(c_2 - c_1) + u \\
 V_2 \frac{dc_2}{dt} &= k_{12}(c_1 - c_2) + k_{23}(c_3 - c_2) - k_0 c_2 \\
 V_3 \frac{dc_3}{dt} &= k_{23}(c_2 - c_3)
 \end{aligned} \tag{1}$$

As output we can choose to study the concentration in either of the compartments, i.e., we can choose between $y_1 = c_1$, $y_2 = c_2$, or $y_3 = c_3$.

Introduce the state vector

$$x = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

The system can then be written in state-space form as

$$\begin{aligned}
 \frac{dx}{dt} &= Ax + Bu \\
 y_i &= C_i x
 \end{aligned} \tag{2}$$

where the system and input matrices are given by

$$A = \begin{pmatrix} \frac{-k_{12}}{V_1} & \frac{k_{12}}{V_1} & 0 \\ \frac{k_{12}}{V_2} & \frac{-k_{12}-k_{23}-k_0}{V_2} & \frac{k_{23}}{V_2} \\ 0 & \frac{k_{23}}{V_3} & \frac{-k_{23}}{V_3} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{V_1} \\ 0 \\ 0 \end{pmatrix}$$

and where the output matrix can be either one of

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Problem 2. We have formulated a linear dynamical model. Give some examples of nonlinearities that might exist in real compartment systems that are not captured by a linear model.

□

3. Examination of the System

We assume the following set of parameters to begin with:

$$\text{Case 1: } V_1 = V_2 = V_3 = k_{12} = k_{23} = k_0 = 1$$

Problem 3. Enter the system in MATLAB as three different state-space models (corresponding to the three measurement signals), see the code skeleton in Appendix A.

Then compute the transfer functions $G_1(s)$ (from u to y_1), $G_2(s)$ (from u to y_2), and $G_3(s)$ (from u to y_3), as well as their poles and zeros. Is the system stable? Asymptotically stable?

Problem 4. Plot the impulse responses of $G_1(s)$, $G_2(s)$, and $G_3(s)$ in one figure. Comment on the dynamic behavior. Does it seem reasonable?

Problem 5. Plot the step responses for $G_1(s)$, $G_2(s)$, and $G_3(s)$ in one figure. Comment on the dynamic behavior. Does it seem reasonable? (Hint: Reason about how the model (1) behaves in equilibrium, i.e., when $\frac{dc_i}{dt} = 0$.)

No Outlet

We shall now study what happens if there is no outlet in the system. Use the following set of parameters:

$$\text{Case 2: } V_1 = V_2 = V_3 = k_{12} = k_{23} = 1, \quad k_0 = 0$$

Problem 6. Compute the new transfer functions $G_1(s)$, $G_2(s)$, and $G_3(s)$, as well as their poles and zeros. Is the system stable? Asymptotically stable?

Problem 7. Plot the impulse responses for $G_1(s)$, $G_2(s)$, $G_3(s)$ in one figure. Comment on the dynamic behavior. Does it seem reasonable?

Faster Exchange Between Compartments 2 and 3

We shall now study what happens if the exchange between compartments 2 and 3 becomes 20 times faster compared to Case 1. Use the following set of parameters:

$$\text{Case 3: } V_1 = V_2 = V_3 = k_{12} = k_0 = 1, \quad k_{23} = 20$$

Problem 8. Compute the new transfer functions $G_1(s)$, $G_2(s)$, and $G_3(s)$, as well as their poles and zeros. Where is the fastest pole located? Compare with problem 3.

Problem 9. Plot the impulse responses of $G_1(s)$, $G_2(s)$, and $G_3(s)$ in one figure. Comment on the dynamic behavior. Does it seem reasonable?

4. Automatic Control

Assume that we want to control the concentration in compartment 3, such that the y_3 approaches the setpoint $r = 1$. To accomplish this, we may apply a feedback proportional to the control error $e = r - y_3$:

$$u = K(r - y_3) \quad (3)$$

Problem 10. Substitute the control law (3) into the model (2) (with $i = 3$) and write down the closed-loop system using matrices. Note that the closed-loop system has the input signal r and the output signal y_3 .

Assume the set of parameters from Case 1 and two different settings for the proportional gain, $K = 1$ and $K = 10$. Enter the resulting two different closed-loop systems in MATLAB and plot their step responses in one figure. Compare and comment the behavior in the two cases.

Finally, examine how large K may become before the closed-loop system goes unstable.

A. Code Skeleton

```
% Define model parameters
V1 = ... ;
V2 = ... ;
V3 = ... ;
k12 = ... ;
k23 = ... ;
k0 = ... ;

% Definiera matricer
A = ... ;
B = ... ;
C1 = ... ;
C2 = ... ;
C3 = ... ;
D = ... ;

% Define state-space models
sys1 = ss(A,B,C1,D);
sys2 = ss(A,B,C2,D);
sys3 = ss(A,B,C3,D);
```

List of useful MATLAB commands (also see Exercise Session 4)

```
help
ss
tf
pole
zero
impulse
step
legend
title
xlabel
ylabel
grid
```

Hint: Both `impulse` and `step` can produce multiple plots at the same time, e.g.

```
>> impulse(sys1,sys2,sys3)
>> step(sys1,sys2,sys3)
```