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Institutionen för
REGLERTEKNIK

Automatic Control, Basic Course (FRTF05)

Exam December 19, 2017, 13:00-18:00
in room F117 of New Main Building, BUAA

Points and grades

All answers must include a clear motivation. The maximal number of points is 25. The maximal number of points is specified for each subproblem.

Betyg 3: minimum 12 points
4: minimum 17 points
5: minimum 22 points

Accepted aids

Mathematical collections of formulae (e.g. TEFYMA); 'Collections of formulae in automatic control'; calculator that is not programmed in advance.

Results

The results will be posted on the course home page and the graded exam will be displayed on Tuesday January 9, in lab C, Dept of Automatic Control. at 12.30-13.00. Thereafter, exams will be archived at the Automatic Control Department in Lund and at BUAA, Beijing, respectively.

Solutions

1. A simple model for an electric generator is given by

$$J\ddot{\theta}(t) = M_d(t) + M(t),$$

where $\theta(t)$ is the angle of the generator shaft, J is the moment of inertia, $M_d(t)$ is the torque driving the generator and $M(t)$ is a torque due to electro-magnetic phenomena. A simplified expression for the torque M is

$$M(t) = -f\dot{\theta}(t) + K\sin(\omega_0 t - \theta(t)),$$

where the first term describes energy losses in the generator, and the second term describes interaction with the rest of the power grid. The sign of $\theta(t) - \omega_0 t$ determines if the generator is delivering or receiving power from the rest of the grid, and this value will be our measurement.

a. Let $x_1(t) = \theta(t) - \omega_0 t$ and show that the system can be written on the state-space form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -b\sin(x_1(t)) - a x_2(t) + u(t), \\ y(t) &= x_1(t), \end{aligned}$$

where the system input is of the form $u(t) = \frac{M_d(t) - c}{d}$. Express a, b, c and d in terms of the non-negative constants f, ω_0, K and J. (0.5 p)

b. Find all stationary points (x_1^0, x_2^0, u^0, y^0) for the system. (0.5 p)

c. Linearize the system at the stationary point corresponding to $x_1^0 = \pi/2$. Is the linearized system asymptotically stable, stable or unstable? (1 p)

Solution

a. We define $x_1 = \theta - \omega_0 t$, $x_2 = \dot{x}_1 = \dot{\theta} - \omega_0$ and get

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\frac{K}{J}\sin(x_1(t)) - \frac{f}{J}x_2(t) + \underbrace{\frac{M_d(t) - f\omega_0}{J}}_{u(t)}, \\ y(t) &= x_1(t). \end{aligned}$$

Comparing coefficients gives

$$a = f/J, \quad b = K/J, \quad c = f\omega_0, \quad d = J.$$

b. In stationarity we have

$$\begin{aligned} 0 &= x_2^0, \\ 0 &= -b\sin(x_1^0) - a x_2^0 + u^0, \\ y^0 &= x_1^0, \end{aligned}$$

which is equivalent to

$$x_2^0 = 0, \quad u^0 = b\sin(x_1^0), \quad y^0 = x_1^0.$$

Thus there are infinitely many stationary points for this system. All stationary points can be expressed as

$$(x_1^0, x_2^0, u^0, y^0) = (\alpha, 0, b\sin(\alpha), \alpha),$$

for some $\alpha \in \mathbb{R}$.

c. The choice $x_1^0 = \pi/2$ corresponds to the stationary point $(x_1^0, x_2^0, u^0, y^0) = (\pi/2, 0, b, \pi/2)$. Computing all partial derivatives and inserting the stationary point yields

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=x^0, u=u^0} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\Bigg|_{x=x^0, u=u^0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}\Bigg|_{x=x^0, u=u^0} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\frac{\partial g}{\partial u}\Bigg|_{x=x^0, u=u^0} = 0,$$

and the linearized system is thus

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}}_{A} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \Delta u(t),$$

$$\Delta y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}.$$

To determine stability, we compute the poles of the system, which are given by the eigenvalues λ of the matrix A:

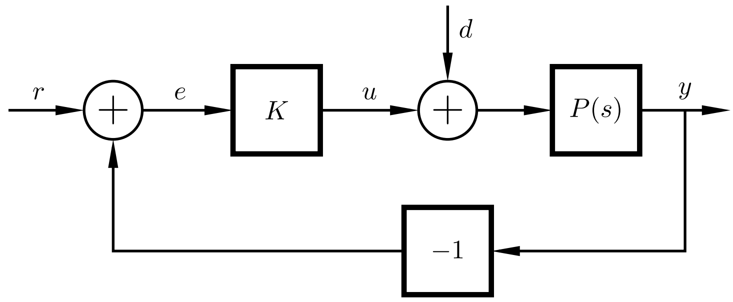
$$\det(\lambda I - A) = 0 \Leftrightarrow \lambda(\lambda + a) = 0.$$

Thus the poles are given by $\lambda_1 = 0$ and $\lambda_2 = -a$. If $a > 0$, we have one pole in the LHP and one pole on the imaginary axis. The system is therefore stable (but **not** asymptotically stable). However, if $a = 0$ there is a double pole at the origin, $\lambda = 0$, which implies an **unstable** system.

2. Consider the feedback system in Figure 1, where a P-controller is used to control the process $P(s)$. The process $P(s)$ has exactly one pole in the origin. Assume that K has been chosen such that the feedback system is asymptotically stable.

a. Consider a unit step reference signal r, i.e.,

$$r(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$



Figur 1 Feedback system in problem 2.

Determine the size of the stationary error. (1 p)

Hint: The process can be written as $P(s) = P^*(s)\frac{1}{s}$, where $P^*(s)$ is a transfer function with no poles or zeros in the origin.

b. Consider a unit step load disturbance d. Determine the size of the stationary error. (1 p)

c. Assume now that $P(s) = 1/s$. Find a new controller (i.e., not necessarily a P-controller) that satisfies:

- The feedback system is asymptotically stable.
- The stationary error is zero when the reference r is a unit step.
- The stationary error is zero when the load disturbance d is a unit step.

Clearly show that your new controller satisfies the specifications. (1 p)

Solution

a. We use the hint and consider $P(s) = P^*(s)\frac{1}{s}$. The transfer function from reference signal r to control error e is

$$G_{r\to e}(s) = \frac{1}{1+P(s)K} = \frac{s}{s+P^*(s)K}.$$

The unit step reference signal has the Laplace transform $R(s) = 1/s$. Since the feedback system is asymptotically stable we can use the final value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sG_{r\to e}(s)\frac{1}{s} = \lim_{s\to 0}\frac{s}{s+P^*(s)K} = 0.$$

Thus the stationary error is zero.

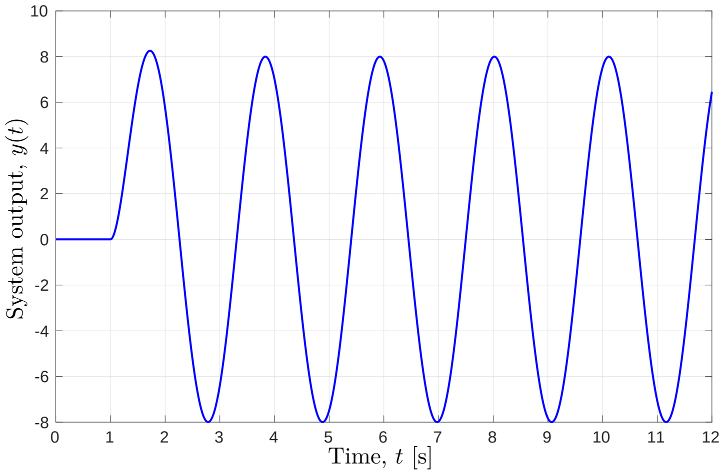
b. The transfer function from disturbance d to control error e is

$$G_{d\to e} = \frac{-P(s)}{1+P(s)K} = \frac{-P^*(s)}{s+P^*(s)K}.$$

The disturbance is $D(s) = \frac{1}{s}$. Since the feedback system is asymptotically stable, we can use the final value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sG_{d\to e}(s)\frac{1}{s} = \lim_{s\to 0}\frac{-P^*(s)}{s+P^*(s)K} = -\frac{1}{K}.$$

Thus the stationary error is $-1/K$.



Figur 2 System response of $G(s)$ in problem 3.

c. A P-controller can be used to achieve an asymptotically stable feedback system, but as seen in **b**. it will not be able to handle load disturbances. To achieve zero stationary error for both reference and disturbance we need to introduce an integrator in the controller. Consider e.g. the PI-controller $K(1 + 1/s)$, which gives the characteristic equation:

$$s^2 + Ks + K = 0,$$

which has roots in the LHP as long as $K > 0$. Thus the closed-loop system is asymptotically stable for $K > 0$. Also, we get zero stationary errors since:

$$G_{r\to e}(s) = \frac{s^2}{s^2 + Ks + K}, \qquad \Longrightarrow G_{r\to e}(0) = 0,$$
$$G_{d\to e}(s) = \frac{-s}{s^2 + Ks + K}, \qquad \Longrightarrow G_{d\to e}(0) = 0.$$

3. A linear system $G(s)$ has at times $t < 0$ the stationary output $y(t) = 0$. At time $t = 0$ we apply the following input signal u to the system:

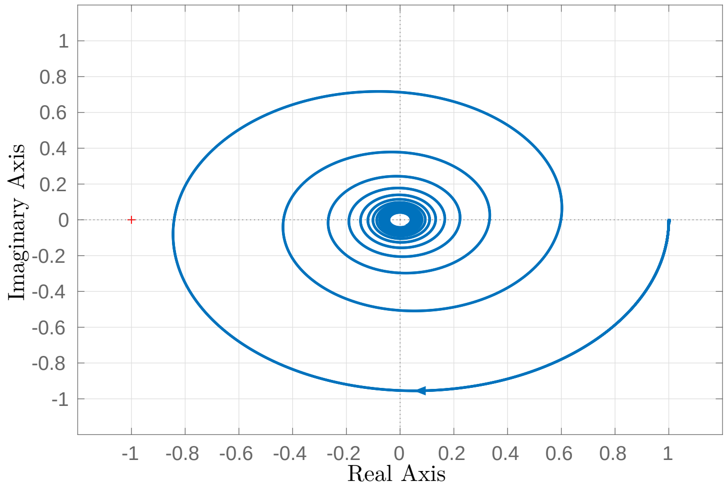
$$u(t) = 10\sin(3t), \quad t \geq 0.$$

The output y of the system for $t \geq 0$ is shown in Figure 2. Also, a Nyquist diagram of $G(s)$ is plotted in Figure 3. Assume that the system is of the form:

$$G(s) = \frac{1}{sT + 1}e^{-Ls}.$$

a. Determine the parameters T and L. (1 p)

b. Sketch the step-response $y(t)$, $t \geq 0$, of $G(s)$. In particular, make sure to mark out L and $L+T$ on the time axis, and the initial and final value of $y(t)$ in your sketch. (1 p)



Figur 3 Nyquist diagram of the system $G(s)$ for problem 3.

c. We wish to introduce feedback to control the system. Can we design a P-controller such that the stationary error is less than 0.3 when making a unit step in the reference signal? (2 p)

Solution

a. The parameters can be determined from the output signal in Figure 2. First we note that the output signal has a delay of 1 s, which immediately gives $L = 1$. Secondly, after the transients have died out, we note that the output has an amplitude of 8, when the input is a sinusoid of frequency 3 rad/s and amplitude 10. Thus we know that:

$$|G(3i)| = \frac{8}{10} = \frac{4}{5}$$

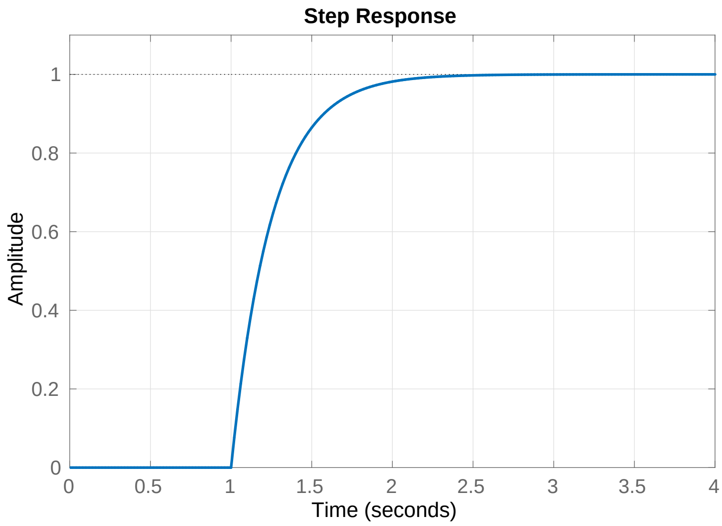
Since we know the form of $G(s)$, we get:

$$|G(3i)| = \frac{1}{\sqrt{9T^2+1}} = \frac{4}{5} \quad \Longrightarrow T = \frac{1}{4},$$

where the solution $T > 0$ is chosen since the system is stable. In summary, $T = 1/4$ and $L = 1$.

b. The system is a stable first-order system with a time-delay. After $L = 1$s, it will rise from the initial value 0 to the final value of 1 (static gain $G(0) = 1$) without overshoot. Also, we have $y(L+T) \approx 0.63$. With this knowledge, we can sketch the step response, see Figure 4. We can also use the inverse Laplace transform to get the time-response explicitly:

$$\mathcal{L}^{-1}[\frac{1}{sT+1}e^{-Ls}\frac{1}{s}] = \begin{cases} 1 - e^{-(t-L)/T}, & t > L \\ 0, \quad t \leq L \end{cases} = \begin{cases} 1 - e^{-4(t-1)}, & t > 1 \\ 0, \quad t \leq 1 \end{cases}$$



Figur 4 Step response of system $G(s)$ for problem 3b.

c. Let the P-controller have the gain K. The closed-loop transfer function from reference signal r to control error e is then given by:

$$G_{r \to e}(s) = \frac{1}{1 + G(s)K}.$$

Assuming that the closed-loop system is stable, the final-value theorem gives the stationary error when the reference signal is a unit step:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = G_{r \to e}(0) = \frac{1}{1 + G(0)K} = \frac{1}{1 + K}.$$

Thus, to get a stationary error smaller than 0.3 we require:

$$\frac{1}{1+K} < 0.3 \quad \Leftrightarrow \quad \frac{7}{3} < K.$$

However, we note in the Nyquist diagram in Figure 3 that the gain margin of $G(s)$ is approximately $1/0.84 \approx 1.2 < 7/3$. Thus the closed-loop system will be unstable for $K > 1.2$ and we can't achieve the specification on stationary error being less than 0.3.

4. A system is given by the state-space model

$$\dot{x} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

a. Calculate the controllability matrix. Is the system controllable? If not, which are the controllable states? (1 p)

b. Find a control law $u = -Lx + l_r r$ so that the closed-loop poles are in $-1, -2$ and -3. (2 p)

c. What restriction do we have to consider when placing the closed loop poles? (1 p)

Solution

a.

$$W_c = [\,B \quad AB \quad AAB\,] = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix does not have full rank (last row is zero), so the system is not controllable.
The controllable states are given by the columns of W_c. It is clear that x_1 and x_2 are controllable, i.e., the controllable states are

$$x = [\,\alpha \quad \beta \quad 0\,]^T\,,$$

for any α, β.

b. With $L = [\,l_1 \quad l_2 \quad l_3\,]$, we have

$$sI - A + BL = \begin{bmatrix} s+1+l_1 & -2+l_2 & l_3 \\ -1 & s & -1 \\ 0 & 0 & s+1 \end{bmatrix}$$

with determinant

$$\begin{aligned} &(s+1+l_1)s(s+1) - (-2+l_2)(-1)(s+1) \\ =&s^3 + s^2(1+l_1+1) + s(1+l_1) + s(-2+l_2) + (-2+l_2) \\ =&s^3 + s^2(2+l_1) + s(l_1+l_2-1) + (-2+l_2) \end{aligned}$$

Equating to

$$\begin{aligned} &(s+1)(s+2)(s+3) \\ =&s^3 + s^2(3+2+1) + s(2+3+6) + 6 \\ =&s^3 + 6s^2 + 11s + 6 \end{aligned}$$

gives $(-2+l_2) = 6 \Rightarrow l_2 = 8$, $(l_1+l_2-1) = 11 \Rightarrow l_1 = 4$. And $2 + l_1 = 6$. So $L = [\,4 \quad 8 \quad l_3\,]$, where l_3 can be anything.

c. Since the system is not controllable, we are not able to place the poles arbitratily. As seen in the first line of the calculations for the determinant, we will always have a pole in -1. This can also be seen in the state space equations for the third state.

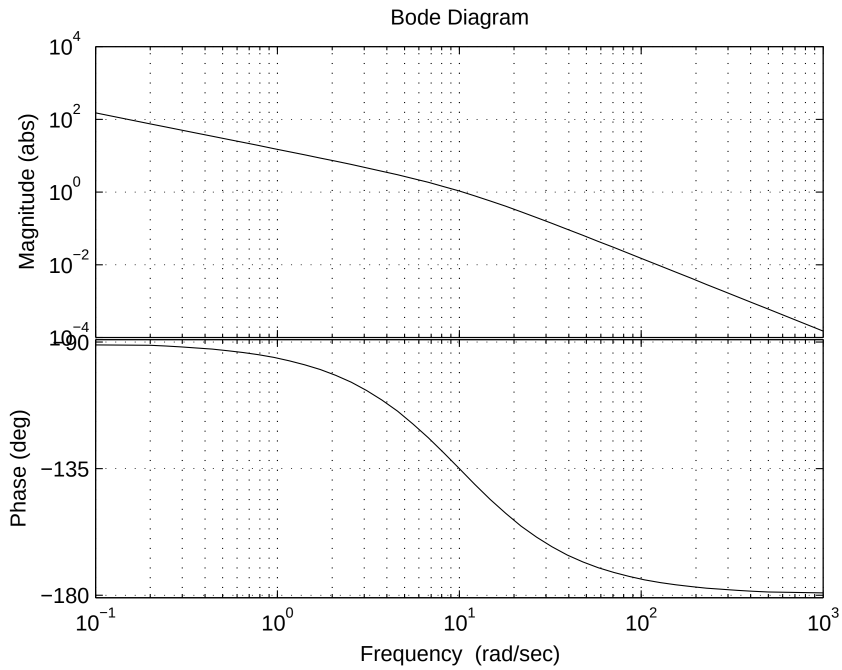
5. A system with transfer function

$$G_P = \frac{15}{s(0.1s+1)}$$

is controlled (in a standard negative feedback loop) by a P–controller with gain $K = 1$. However, the closed-loop system is considered to be too slow and you want to make it twice as fast without getting a decreased phase margin. Dimension a phase-advanced link (a so-called lead link)

$$G_K = K_K N \frac{s+b}{s+bN}$$

such that the cross-over frequency ω_c increases a factor 2 without decreasing the phase margin. The Bode diagram for G_P is shown in Figure 5. (2.5 p)



Figur 5 Bode diagram for Problem 5

Solution From the Bode diagram the current cross-over frequency can be found to be $\omega_c \approx 10 rad/s$. As alternative it can also be calculated as

$$\begin{aligned}
\arg G(i\omega_c) &= \arg 15 - (arg(i\omega_c) + arg(i\omega_c 0.1 + 1)) \\
\arg G(i\omega_c) &= 0 - 90^\circ - \arctan(0.1\omega_c) \\
\arg G(i\omega_c) &\approx -135^\circ
\end{aligned}$$

and

$$\phi_m = 180^\circ + \arg G(i\omega_c) \approx 45^\circ$$

Thus, we would like to double $\omega_c = 10$ to $\omega_c^* = 20$ while preserving the phase margin. From the Bode diagram (or with corresponding calculations as above) we find that we need to increase the phase approximately 18° at ω_c^* to keep the same phase margin. From the collection of formulae one can see the $N = 2$ is enough.

To get the maximum phase increase exactly at ω_c^* you put

$$b\sqrt{N} = \omega_c^*$$

which gives $b = 14$. Now one should adjust the total gain so that the cross-over frequency becomes ω_c^*.

$$|G_K(i\omega_c^*)G_P(i\omega_c^*)| = 1$$

In this expression everything is know except K_K. Solving the equation gives $K_K = 2$ and the phase advanced link (the lead link) becomes

$$G_K = K_K N \frac{s+b}{s+bN} = 4\frac{s+14}{s+28}$$

6. In most countries around the world, the frequency in the power sockets is $50Hz$ i.e., 100π rad/s. Measuring this would result in a signal of the form

$$y(t) = V\sin(\omega t) + n(t),$$

where $\omega = 100\pi$, V is the peak voltage, and n is measurement noise.

This signal can be modeled as being generated by the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 100\pi \\ -100\pi & 0 \end{bmatrix} x(t)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + n(t).$$

To have a good estimate of the peak voltage V at all times, it is nessesary to know both states in the system.

a. Design a Kalman filter to estimate the two states. Place both of the two poles in -100π and determine the corresponding vector K. (1 p)

b. The effect of the noise n on the error of the estimate $\tilde{x} = x - \hat{x}$ can be written as

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) - Kn(t),$$

where A is the system matrix above, and K the Kalman gain vector. We do not want the noise to affect the estimation error too much. If we approximately know the frequency of the noise n, describe how we then should reason on where to place the poles for the Kalman filter with respect to the noise frequency. (You do not need to calculate any value for K in this subproblem). (1 p)

Solution

a. We have

$$sI - (A - KC) = \begin{bmatrix} s + k_1 & -100\pi \\ 100\pi + k_2 & s \end{bmatrix}$$

with characteristic equation $s^2 + sk_1 + 100^2\pi^2 + 100\pi k_2$.
Equating with $(s + 100\pi)^2 = s^2 + 200\pi s + 100^2\pi^2$ gives $k_1 = 200\pi, k_2 = 0$.

b. We want the transfer function from n to $\tilde{x}$ to be small. We should therefore make sure to place the two poles at frequencies well below the frequency of the noise.

7. You are working as a consultant and are assigned to control a process with dynamics

$$P(s) = \frac{1}{s-1}.$$

During a design meeting, your colleague suggests the controller

$$C(s) = \frac{s-1}{s+1}.$$

a. Will the closed-loop system from reference to process output be stable with the proposed controller? (1 p)

b. The proposed controller has a zero at $s = 1$, which cancels the process pole. What practical problem could this type of unstable pole-zero cancellation cause? Explain, motivated by calculations for the particular example. (1 p)

c. Design a controller $C(s)$, which stabilizes the transfer function from reference to process output, without cancelling the unstable process pole. (1 p)

Solution

a. The open-loop transfer function becomes

$$G_o(s) = P(s)C(s) = \frac{1}{s+1},$$

yielding the closed-loop transfer function

$$G_{yr}(s) = \frac{G_o(s)}{1+G_o(s)} = \frac{1}{s+2}.$$

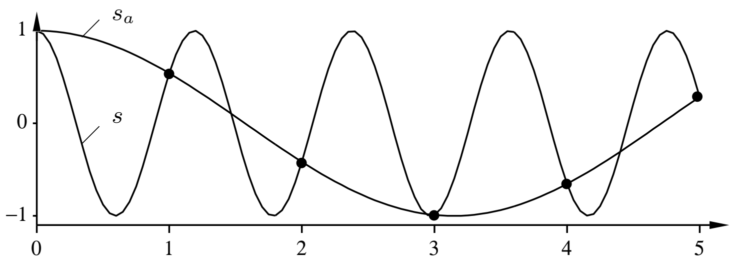
The closed-loop system from reference r to process output y is asymptotically stable since the only pole of $G_{yr}(s)$ lies at $s = -1$, which is strictly in the left half plane. (However, the system exhibits poor reference tracking, since $G_{yr}(0) = 1/2 \neq 1$.)

b. By cancelling the unstable process zero with a controller pole, we make the zero dynamics uncontrollable from the reference r. The zero dynamics are, however, still observable from the process output y and (obviously) controllable from the control signal u. The transfer function from control signal (process input) u to process output y becomes

$$G_{yu}(s) = \frac{P(s)}{1+G_o(s)} = \frac{s+1}{(s-1)(s+2)}.$$

It has an unstable (right half plane) pole at $s = 1$, meaning that an additive load disturbance step, introduced at the process input, would drive the process output y unstable.

Another problem is that small deviations in process or controller dynamics could, for instance $P(s) = 1/(s-1-\epsilon)$ with $\epsilon \neq 0$ results in the closed-loop from r to y becoming unstable. Such small deviations are almost always impossible to avoid in practice, due to for example component manufacturing tolerances.



Figur 6 Illustration of the aliasing phenomenon. The figure shows the actual signal s and its alias s_a.

c. There are several approaches to stabilizing the dynamics. One approach is to introduce the state $x = y$, resulting in the state space representation $\dot{x} = x + u$. Introducing the state feedback $u = -kx$ yields the closed-loop dynamics $\dot{x} = (1-k)x$. Choosing $k > 1$ results in the closed loop system matrix having strictly negative eigenvalue, making the closed-loop asymptotically stable.

Since $y = x$, our state feedback controller is in fact a P controller $u = -ky$ with gain k. No pole-zero cancellation takes place since the controller lacks zeros.

Comment: Since we have full access to the state through y, we can make a full state feedback design without introducing an observer. The system is furthermore controllable, since the controllability matrix $W_c = B = 1 \neq 0$ has full rank. This means that we can place the closed-loop controller pole arbitrarily.

8. Today, almost all controllers are implemented in computers, i.e. the signals must be discretized. In order to do this, a sampling frequency, ω_s, is selected. The sampling frequency introduces an upper bound, referred to as the Nyquist frequency ω_N, on how high frequencies that are visible for the controller.

$$\omega_N = \frac{\omega_s}{2}$$

In other words we must keep the sampling frequency at least twice as high as the highest frequency relevant to the controller, if not we will have the effect of aliasing. Explain the aliasing phenomena by using a figure. (1 p)

Solution The phenomena is visualized in the figure fig:aliasing. Due to the relatively low sampling frequency, the controller will not obtain the signal s, but rather the signal s_a, which has a significantly lower frequency.

9. A common structure for PID-controllers is:

$$U = K\left((bR - Y) + \frac{1}{sT_i}E - \frac{sT_d}{1 + sT_d/N}Y\right)$$

Give the formula for a corresponding discretized PID controller, and show how you determined the discretized formula. (1.5 p)

Solution

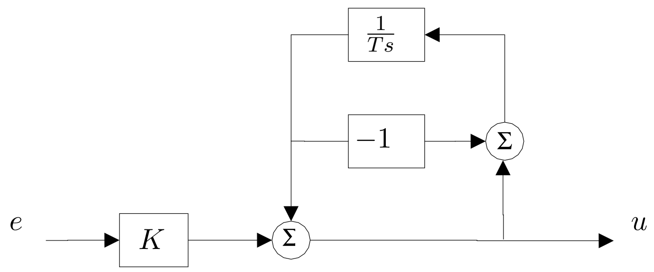
a. The discrete control signal is given by

$$u(kh) = P(kh) + I(kh) + D(kh)$$

i.e., a sum of the proportional, integral and derivative parts. The sampling interval is denoted h and k is an integer.

- P-part:
$$P(kh) = K(br(kh) - y(kh))$$
- I-part: Approximate the integral by replacing it with a sum.
$$I(kh) = I(kh - h) + \frac{Kh}{T_i} e(kh)$$
- D-part: Approximate the derivatives by replacing them by differences.
$$D(kh) = \frac{T_d}{T_d + Nh} D(kh - h) - \frac{KT_d N}{T_d + Nh} (y(kh) - y(kh - h))$$

10. The block diagram in Figure 7 shows a controller with an "automatic offset adjustment". Calculate the transfer function $G_{e\to u}(s)$ for the controller. What "familiar controller" is it? (2 p)



Figur 7 Block diagram for the controller in Problem 10.

Solution The signal u is given by the equation

$$
\begin{aligned}
& U(s) = KE(s) + \left(\frac{\frac{1}{Ts}}{1+\frac{1}{Ts}} \right) U(s) \\
\iff \quad & U(s) = KE(s) + \left(\frac{1}{Ts+1} \right) U(s) \\
\iff \quad & U(s) \left(1 - \frac{1}{Ts+1} \right) = KE(s) \\
\iff \quad & U(s) = K \left(\frac{Ts+1}{Ts} \right) E(s)
\end{aligned}
$$

The controller is a PI–controller.

Good luck!