Transient Response, Step Response Analysis

Automatic Control, Basic Course, Lecture 3

November 5, 2019

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Content

Transient Response

Solution to State Space Equation

Given a system on state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The solution, $y(t)$, is then given by

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Initial state, uninteresting except when the controller is initialized

Weighted integral of the control signal, interesting part

Direct term, often neglectable in practical systems

Impulse Response

Shows how the system responds when the input is a short pulse, i.e., a Dirac function

$$u(t) = \delta(t)$$

The Laplace transformation is

$$U(s) = \int_0^\infty e^{-st}\delta(t)\mathrm{d}t = 1$$

Hence

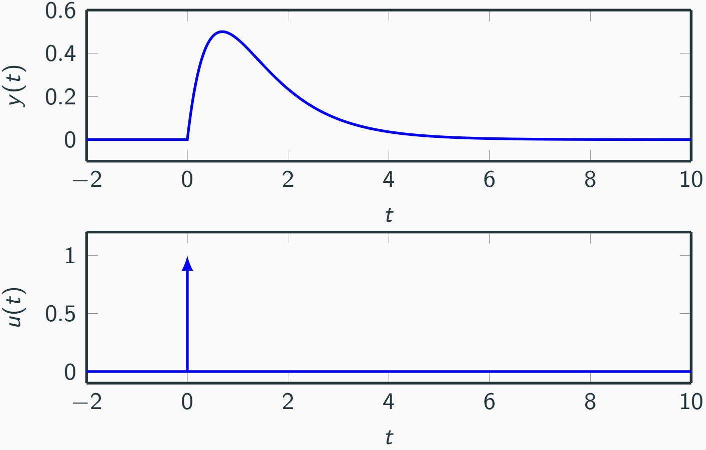
$$Y(s) = G(s)U(s) = G(s)$$

Not so common in technological applications, can we think of other applications?

Example - Impulse Response

Let the transfer function of the system be:

$$G(s) = \frac{2}{s^2 + 3s + 2}$$



Step Response

Shows how the system responds when the input is a step, i.e.,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The Laplace transformation is

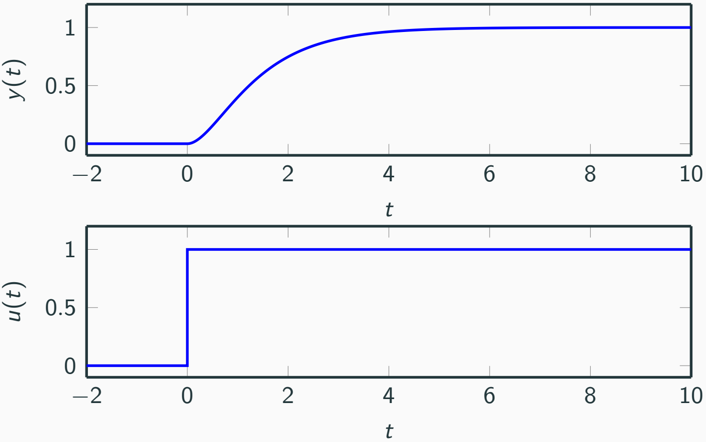
$$U(s) = \int_0^\infty e^{-st} u(t) \mathrm{d}t = \int_0^\infty e^{-st} \mathrm{d}t = -\frac{1}{s} \left[e^{-st} \right]_0^\infty = \frac{1}{s}$$

Very common in technological applications

Example - Step Response

Let the transfer function of the system be:

$$G(s) = \frac{2}{s^2 + 3s + 2}$$



Step Response Analysis

Step Response

From the last lecture, we know that if the input $u(t)$ is a **step**, then the output in the Laplace domain is

$$Y(s) = G(s)U(s) = G(s)\frac{\mathbf{1}}{\mathbf{s}}$$

It is possible to do an inverse transform of $Y(s)$ to get $y(t)$, but is it possible to claim things about $y(t)$ by only studying $Y(s)$?

We will study **how the poles affects the step response**. (The zeros will be discussed later).

Initial and Final Value Theorem

Let $F(s)$ be the Laplace transformation of $f(t)$, i.e., $F(s) = \mathcal{L}(f(t))(s)$.
Given that the limits below exist[1], it holds that:

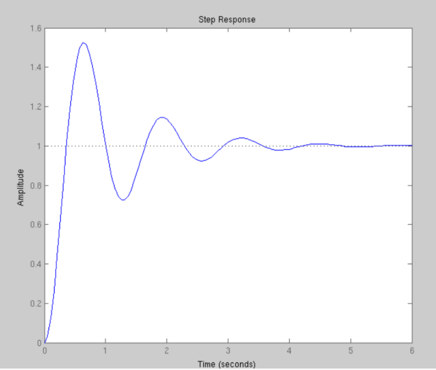
Initial value theorem	$\lim_{t\to 0} f(t) = \lim_{s\to +\infty} sF(s)$
Final value theorem	$\lim_{t\to +\infty} f(t) = \lim_{s\to 0} sF(s)$

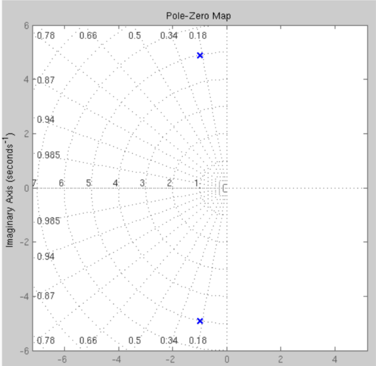
For a step response we have that:

$$\lim_{t\to +\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

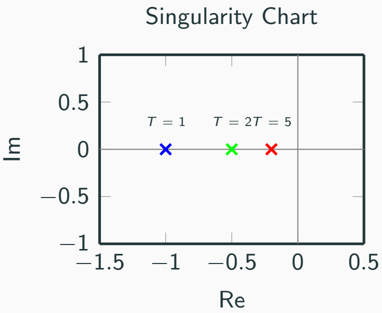
[1] Q: When can we NOT apply the Final value theorem?

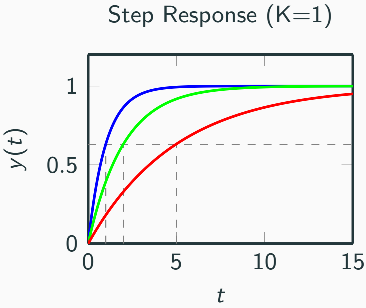
```
>> s=tf('s'); % enables to use s as transfer fcn
>> z=0.2; w0=5;
>> G= w0^2 / (s^2 + 2*z*w0*s + w0^2 )
>> step(G)
>>
>> pzmap(G) % pole-zero map
```





First Order System





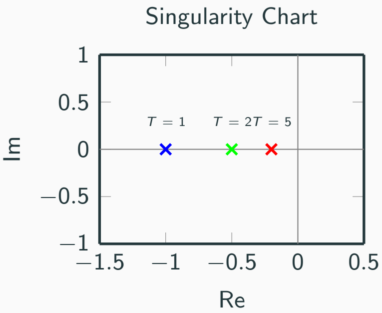
$$G(s) = \frac{K}{1 + sT}$$

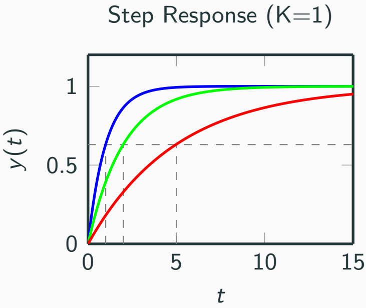
One pole in $s = -1/T$

Step response:

$$Y(s) = G(s)\frac{1}{s} = \frac{K}{s(1+sT)} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = K\left(1 - e^{-t/T}\right), \ \mathbf{t \geq 0}$$

First Order System



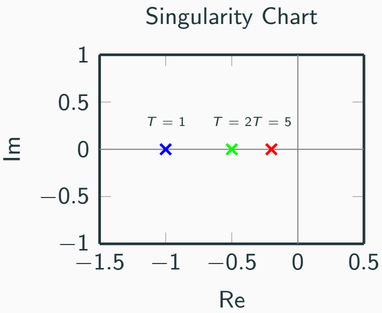


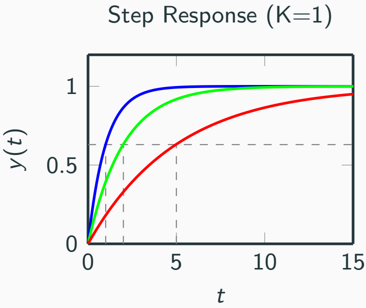
$$G(s) = \frac{K}{1 + sT}$$

Final value:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \cdot \frac{K}{s(1 + sT)} = K$$

First Order System





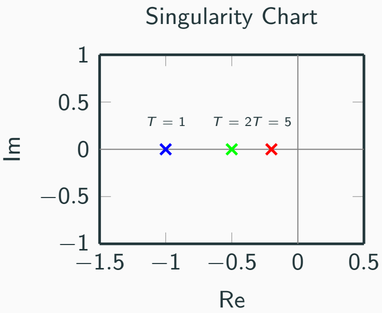
$$G(s) = \frac{K}{1 + sT}$$

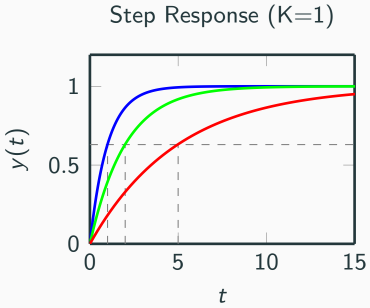
T is called the time-constant:

$$y(T) = K(1 - e^{-T/T}) = K(1 - e^{-1}) \approx 0.63K$$

i.e., T is the time it takes for the step response to reach 63% of its final value

First Order System



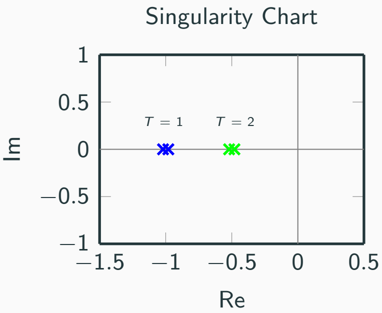


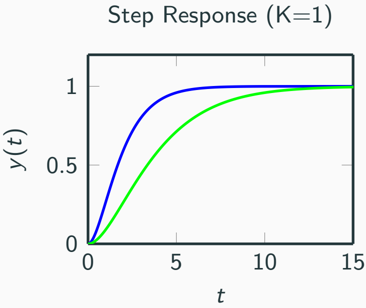
$$G(s) = \frac{K}{1 + sT}$$

Derivative at zero:

$$\lim_{t \to 0} \dot{y}(t) = \lim_{s \to +\infty} s \cdot sY(s) = \lim_{s \to +\infty} \frac{s^2 K}{s(1 + sT)} = \frac{K}{T}$$

Second Order System With Real Poles



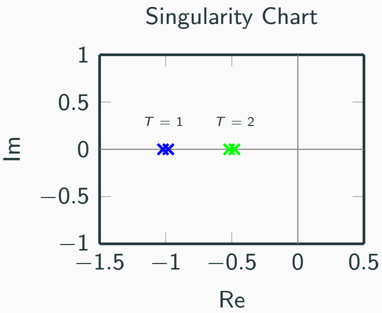


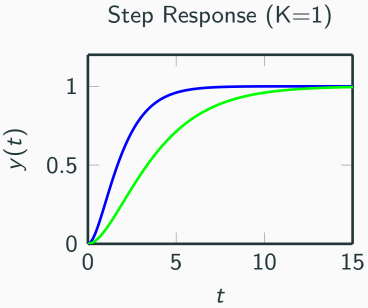
$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Poles in $s = -1/T_1$ and $s = -1/T_2$. Step response:

$$y(t) = \begin{cases} K\left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right), \mathbf{t \geq 0} & T_1 \neq T_2 \\ K\left(1 - e^{-t/T} - \frac{t}{T} e^{-t/T}\right), \mathbf{t \geq 0} & T_1 = T_2 = T \end{cases}$$

Second Order System With Real Poles



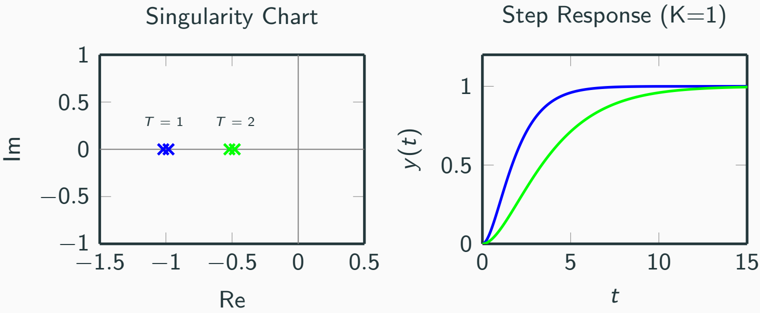


$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Final value:

$$\lim_{t \to +\infty} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{sK}{s(1 + sT_1)(1 + sT_2)} = K$$

Second Order System With Real Poles



$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Derivative at zero:

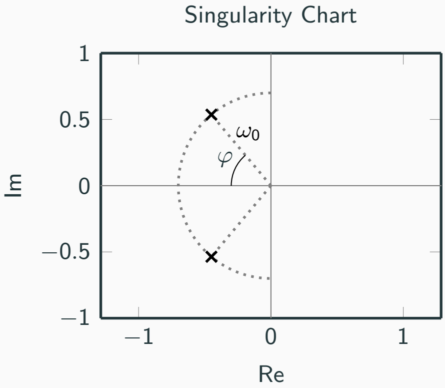
$$\lim_{t \to 0} \dot{y}(t) = \lim_{s \to +\infty} s \cdot sY(s) = \lim_{s \to +\infty} \frac{s^2 K}{s(1 + sT_1)(1 + sT_2)} = 0$$

Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Relative damping ζ, related to the angle φ

$$\zeta = \cos(\varphi)$$



Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Inverse transformation for step response yields:

$$\begin{aligned} y(t) &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\omega_0\sqrt{1-\zeta^2}\,t + \arccos\zeta\right)\right) \\ &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\omega_0\sqrt{1-\zeta^2}\,t + \arcsin(\sqrt{1-\zeta^2})\right)\right), \mathbf{t \geq 0} \end{aligned}$$

Second Order System With Complex Poles

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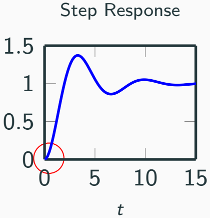
Exercise: Check of correct starting point of step response.

$$\begin{aligned} y(0) &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{0} \sin\left(\omega_0\sqrt{1-\zeta^2}0 + \arcsin(\sqrt{1-\zeta^2})\right)\right) \\ &= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} \cdot \sqrt{1-\zeta^2}\right) \\ &= 0 \end{aligned}$$



Second Order System With Complex Poles

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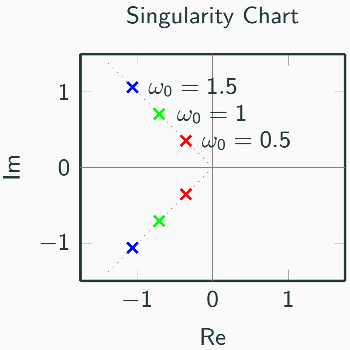
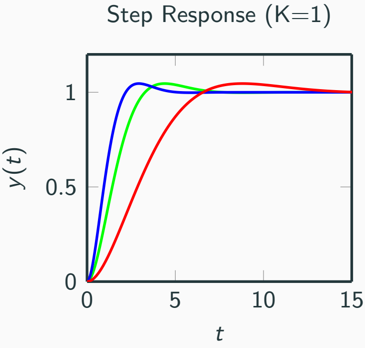
Changing fq ω_0



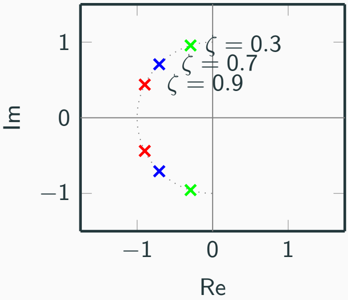


Second Order System With Complex Poles

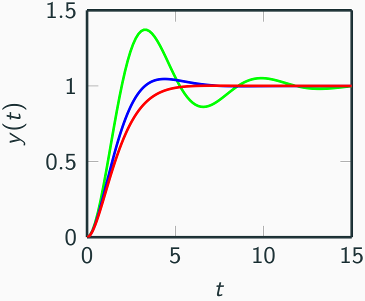
$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Changing damping ζ

Singularity Chart



Step Response (K=1)



Content

This lecture

1. Transient Response

2. Step Response Analysis

Next lecture

- Frequency Analysis