LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Automatic Control, Basic Course (FRTF05)

Exam January 17, 2025, 14.00–19.00

Point and grades

All solutions must include a clear motivation. The total number of points is 50. The maximum number of points is specified for each subproblem. Please note! Only integer points will be rewarded during grading. Grade limits:

Grade 3: 24 points
4: 34 points
5: 44 points

Allowed Aid

Standard mathematical tables like TEFYMA. *Formelsamling i reglerteknik / Collection of Formulae*. Pocket calculator without preprogramming.

Results

The result is announced via LADOK. Time and locations for the exam display will be posted on the homepage and in Canvas.

Instructions

Only write on one side of each sheet of paper.

Start each new problem 1–7 on a new sheet of paper. Solutions to subproblems (a,b,c, ...) may be given on the same paper though.

Good luck!

Solutions to Automatic Control Exam January 17, 2025

1. This problem deals with PID control of the double-tank process.

a. Write down the transfer function of the (ideal) PID controller. (2 p)

b. You are controlling a double-tank process with a PID controller and are not happy with the result—the lower tank level is oscillating too much. Which of the following choices is/are likely to reduce the oscillations?

- Increase K
- Decrease T_i
- Increase T_d

(3 p)

c. The D-part can cause problems if there are step changes in the reference signal. Briefly describe what the problem is and how the controller can be modified to deal with the problem. (2 p)

d. You invest in an extra sensor that can measure an external disturbance flow that enters the *upper* tank. Explain how the control law should be modified to compensate for this disturbance. Draw a block diagram that explains your solution. (3 p)

Solution

a. $G_R(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$

b.
- Increasing K typically means a faster system but worse stability margins.
- Decreasing T_i means more integral action, which usually means worse stability margins.
- Increasing T_d increases the damping, which usually means better stability margins.

The only correct answer is hence to increase T_d. (Note that the effect of changing K and T_i must be commented to receive full point)

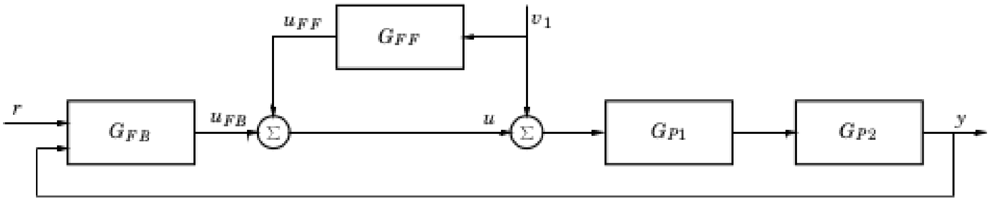
c. The derivative part is given by

$$D(t) = KT_d \frac{de(t)}{dt} = KT_d\left(\frac{dr(t)}{dt} - \frac{dy(t)}{dt}\right)$$

If $r(t)$ changes as a step, then $D(t)$ will be unlimited. A simple solution is to let the derivative part act only on the output:

$$D(t) = -KT_d \frac{dy(t)}{dt}$$

d. If the measured disturbance flow has the same units as the control signal, the controller should simply subtract the disturbance flow from the control signal. In the block diagram below, this would mean chosing $G_{FF} = -1$ to eliminate the influence of the disturbance v_1 entering the upper tank G_{P1}.



2. The population dynamics of two species of fish is given by

$$\dot{x}_1 = x_1(40 - x_1) - 2x_1x_2$$
$$\dot{x}_2 = x_1x_2 - 20x_2$$

a. Find the three stationary points of the system. (3 p)

b. Linearize the system around one of the stationary points you found in **a**. (4 p)

c. Determine the stability of the linearized system you found in **b**. (2 p)

Solution

a. We have

$$\begin{aligned}\dot{x}_1 &= x_1(40 - x_1) - 2x_1x_2 &&= f_1(x_1, x_2)\\ \dot{x}_2 &= x_1x_2 - 20x_2 &&= f_2(x_1, x_2)\end{aligned}$$

In a stationary point (x_1^0, x_2^0), all time derivatives should be zero. We have

$$\begin{aligned}0 &= x_1^0(40 - x_1^0) - 2x_1^0x_2^0\\ 0 &= x_1^0x_2^0 - 20x_2^0\end{aligned}$$

with the solutions $(x_1^0, x_2^0) = (0, 0)$, $(x_1^0, x_2^0) = (40, 0)$, and $(x_1^0, x_2^0) = (20, 10)$.

b. Compute the partial derivatives:

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 40 - 2x_1 - 2x_2 & -2x_1 \\ x_2 & x_1 - 20 \end{pmatrix}$$

Evaluate in a stationary point. The three possible solutions are:

$$\frac{\partial f}{\partial x}(0, 0) = \begin{pmatrix} 40 & 0 \\ 0 & -20 \end{pmatrix} = A_1$$
$$\frac{\partial f}{\partial x}(40, 0) = \begin{pmatrix} -40 & -80 \\ 0 & 20 \end{pmatrix} = A_2$$
$$\frac{\partial f}{\partial x}(20, 10) = \begin{pmatrix} -20 & -40 \\ 10 & 0 \end{pmatrix} = A_3$$

After a change in variables, $\Delta x_1 = x_1 - x_1^0$, $\Delta x_2 = x_2 - x_2^0$, we get the linearized system

$$\Delta\dot{x} = A_i\,\Delta x$$

for one of the A matrices found above.

c. Stability is determined by the eigenvalues of the A matrix. For the diagonal A_1 and the triangular A_2, we immediately see that one eigenvalue is positive (40 and 20, respectively). For A_3, we get the characteristic polynomial

$$\det(sI - A_3) = \det \begin{pmatrix} s-20 & 40 \\ -10 & s \end{pmatrix} = s^2 - 20s + 40$$

Since one coefficient is negative, the linearized system is unstable.

3. You have designed a state feedback control law

$$u = -Kx + k_r r = -x_1 - 2x_2 + 2r$$

for the process

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} c_1 & c_2 \end{pmatrix} x$$

a. Where are the poles of the closed-loop system placed? (2 p)

b. For which values of c_1 and c_2 is the system *not* observable? (2 p)

c. Design an observer (Kalman filter) for the system assuming $c_1 = 1$ and $c_2 = 0$. Motivate any necessary design choices. (4 p)

Solution

a. The characteristic polynomial of the closed-loop system is given by

$$\det(sI - A + BK) = \begin{vmatrix} s & -1 \\ 2 + 2k_1 & s + 2k_2 \end{vmatrix} = s^2 + 4s + 4 = (s+2)^2$$

Both poles of the closed-loop system are hence located in $s = -2$.

b. The observability matrix is

$$W_0 = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ -2c_2 & c_1 \end{pmatrix}$$

By inspection, we note that the rows are linearly indepdendent unless both c_1 and c_2 are zero. Alternatively, computing the determinant,

$$\det W_0 = c_1^2 + 2c_2^2,$$

we note that the determinant can only be zero if both c_1 and c_2 are zero. The system is thus not observable only when $c_1 = c_2 = 0$.

c. We should design an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

As a rule of thumb, the poles of the observer should be twice as fast as the poles of the state feedback. (Selecting any other poles that are faster than the closed-loop poles in **a** is also fine.) The characteristic polynomial of the observer is given by

$$\det(sI - A + LC) = \begin{vmatrix} s + l_1 & -1 \\ 2 + l_2 & s \end{vmatrix} = s^2 + l_1 s + 2 + l_2$$

Comparing with a desired characteristic polynomial of $(s+4)^2 = s^2 + 8s + 16$, we obtain $l_1 = 8$ and $l_2 = 14$.

4. Figure 1 shows the pole-zero maps of four systems, and Figure 2 shows the step responses of the same systems, but not necessarily in the same order. Match the pole-zero maps 1–4 with the corresponding step responses A–D. Motivate! (4 p)

Solution

Since all poles are imaginary, the transfer function of each system can be written in the form

$$G(s) = \frac{\sim}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Larger ω_0 gives faster step response, and poles further away from the origin. Larger ζ gives a more damped step response, and smaller angle between the poles and the negative real axis. Pole–zero maps 1 and 3 have a smaller ζ and only differ by ω_0 (3 being faster than 1). Pole–zero maps 2 and 4 have a larger ζ and only differ by ω_0 (4 being faster than 2).

In conclusion, the correct matching is 1–D, 2–C, 3–B, 4–A.

5. A controller structure known as "mid-ranging control" is shown in Figure 3.

a. Calculate the transfer function from r to y. (3 p)

b. Assume that $G_{P1}(s) = \dfrac{1}{s-1}$, $G_{P2}(s) = \dfrac{1}{s+2}$, $G_{R1}(s) = 1$, and $G_{R2}(s) = \dfrac{1}{s}$. Calculate the stationary value of y, if it exists, when $r(t) = 1$ and $u_{\mathrm{ref}}(t) = 0$. (3 p)

Solution

a. Setting $u_{\mathrm{ref}} = 0$, we obtain

$$Y = (G_{P1} + G_{P2}G_{R2})G_{R1}(R - Y)$$

Solve for Y:

$$Y = \frac{(G_{P1} + G_{P2}G_{R2})G_{R1}}{1 + (G_{P1} + G_{P2}G_{R2})G_{R1}} R$$

b. Inserting $R(s) = \frac{1}{s}$, $G_{P1} = \frac{1}{s-1}$, $G_{P2} = \frac{1}{s+2}$, $G_{R1} = 1$, and $G_{R2} = \frac{1}{s}$ we get

$$Y = \frac{\left(\frac{1}{s-1} + \frac{1}{s+2}\frac{1}{s}\right)}{1 + \left(\frac{1}{s-1} + \frac{1}{s+2}\frac{1}{s}\right)} \cdot \frac{1}{s} = \frac{s^2 + 3s - 1}{s^3 + 2s^2 + s - 1} \cdot \frac{1}{s}$$

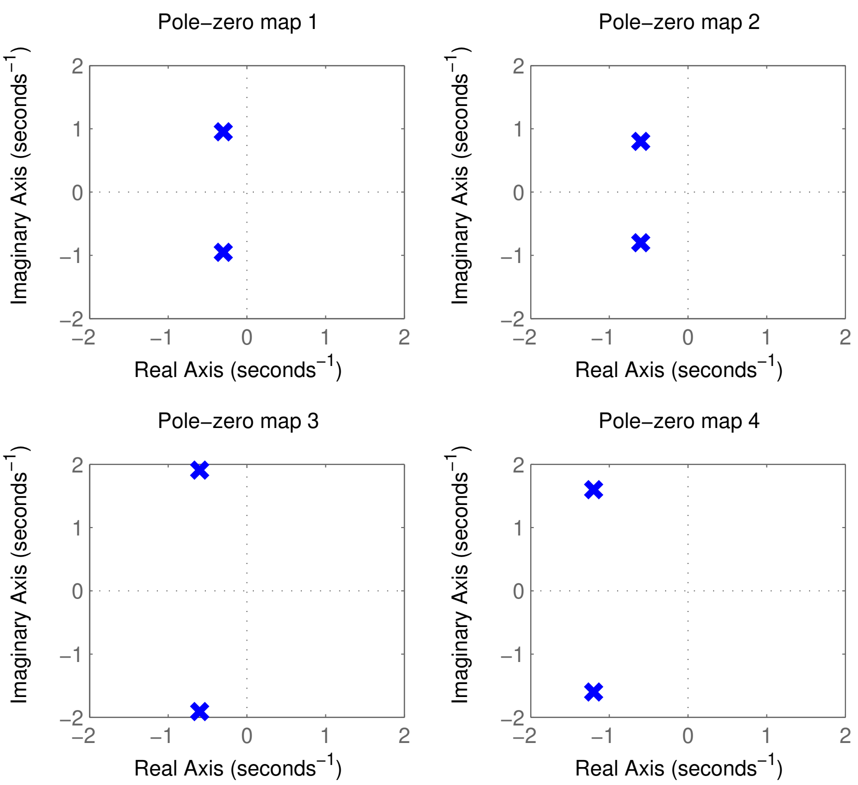


Figure 1 Pole-zero maps for Problem 4

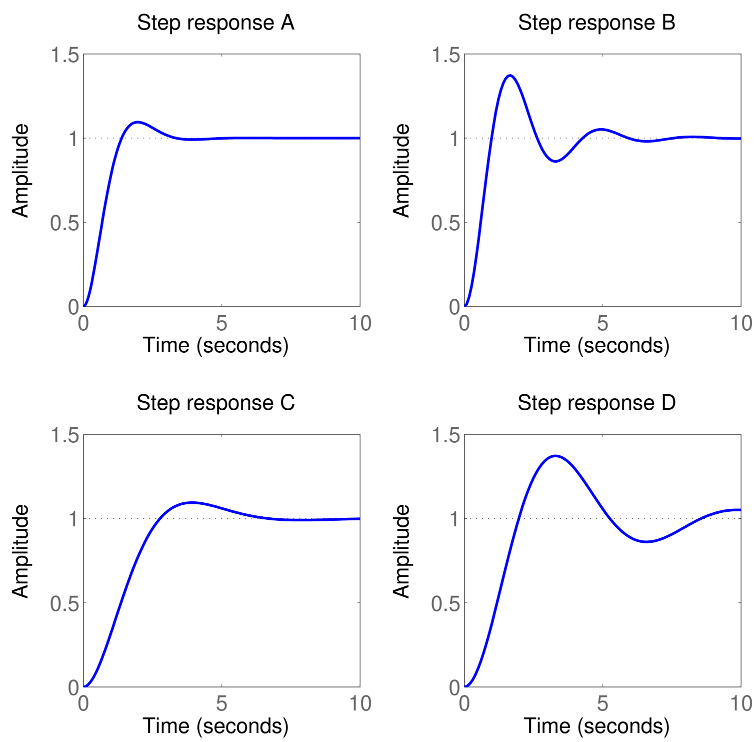


Figure 2 Step responses for Problem 4

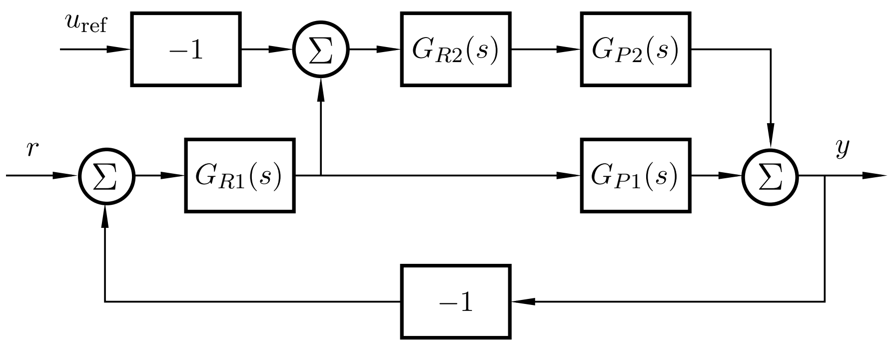


Figure 3 The controller structure in Problem 5.

Attempt to apply the final-value theorem:

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s^2 + 3s - 1}{s^3 + 2s^2 + s - 1}$$

Here we note that $sY(s)$ is unstable (since one coefficient in the denominator is negative). Hence, no stationary value exists.

6. By making frequency response experiments on a mechanical system, you have experimentally obtained the Bode magnitude plot shown in Figure 4.

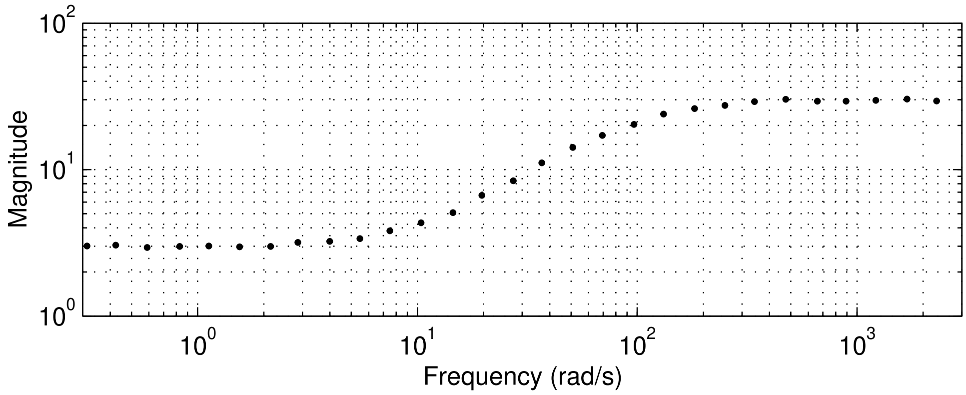


Figure 4 Experimentally obtained Bode magnitude plot in Problem 6

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a. Estimate the transfer function of the system. Assume that the transfer function does not contain any poles or zeros in the right-half plane and no time delays. (4 p)

b. Sketch the Bode phase plot of the system. (Only the low and high frequency asympototes of the phase plot need to be completely accurate.) (2 p)

Solution

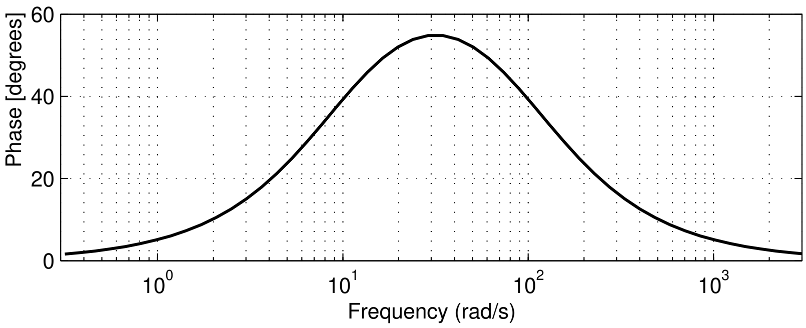
a. By studying the asymptotes of the gain curve and the estimating corner frequencies to about $\omega = 10$ and $\omega = 100$ respectively, we deduce that the transfer function has a zero in $s = -10$ and a pole in $s = -100$. The transfer function can be written

$$G(s) = \frac{K(s+10)}{s+100}$$

where the gain K remains to be determined. By studying the low-frequency asymptote, we see that the system has the static gain $G(0) = 3$. We hence get

$$G(s) = \frac{30(s+10)}{s+100}$$

b. The low-frequency asymptote of the gain curve has the slope 0, hence the phase curve starts at 0° for small ω. It then increases by 90° around the location of the zero at $\omega = 10$ and then decreases by 90° around the location of the pole at $\omega = 100$, finishing at 0° again. The maximum phase lead will be smaller than 90°, but the exact value need not be computed. (Noting that the system is a lead link with $N = 10$, it is possible to deduce that the maximum phase lead is 55°.) The phase curve is shown below.



7. You have designed a controller $G_R(s)$ for some process $G_P(s)$. The Bode plot of the open-loop system $G_0(s) = G_P(s)G_R(s)$ is shown in Figure 5. You have simulated the closed-loop system and are happy with the resulting speed and robustness.

a. Unfortunately, due to a slow Internet connection, the implementation of the controller introduces a delay of 32 ms in the control loop, causing the performance to deteriorate. Design a compensation link $G_K(s)$ such that the original cross-over frequency and phase margin are recovered. (5 p)

b. Having recovered the original cross-over frequency and phase margin using the compensator, is the compensated system as good as the original one? Or are there some other performance differences? (2 p)

Solution

a. From the Bode plot, we see that the original phase margin is $\varphi_m = 45°$ and that the original cross-over frequency is $\omega_c = 20$ rad/s. A loop delay e^{-sL}, where $L = 0.032$, will decrease the phase margin by

$$\omega_c L = 0.64 \text{ rad} = 37°$$

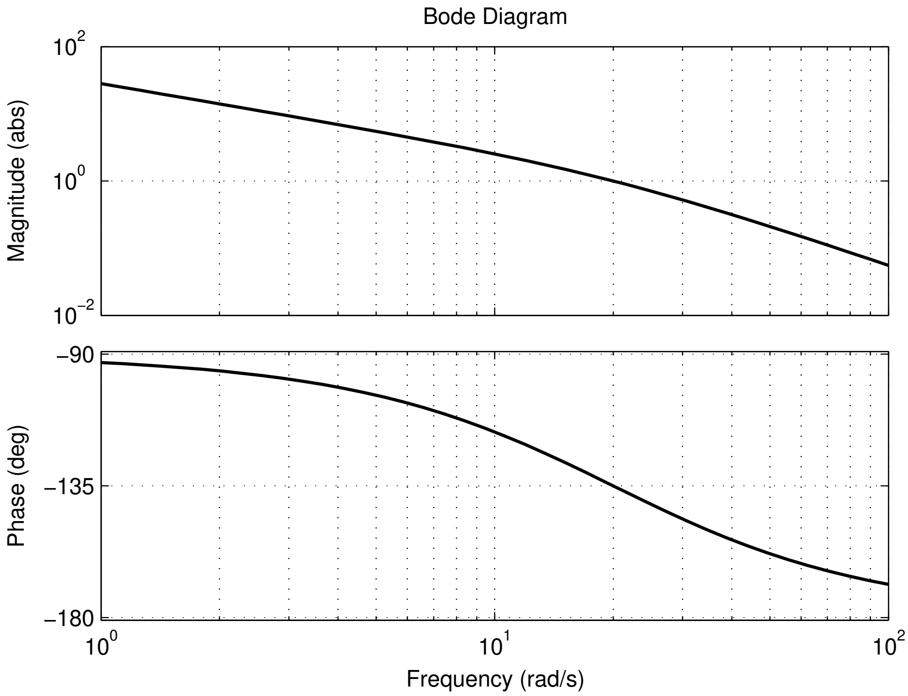


Figure 5 The open-loop Bode diagram in Problem 7

To recover the phase loss, we can use a lead link

$$G_K(s) = K_K N \frac{s+b}{s+bN}$$

with $N = 4$ (taken from the graph in the Collection of Formulae). The parameter b is given by

$$b = \frac{\omega_c}{\sqrt{N}} = 10$$

To get the correct cross-over frequency, it should hold that

$$\underbrace{G_K(i\omega_c)}_{=K_K\sqrt{N}}\underbrace{G_0(i\omega_c)}_{=1} = 1$$

which implies $K_K = \frac{1}{\sqrt{N}} = 0.5$. The final link is hence

$$G_K(s) = 2\frac{s+10}{s+40}$$

b. Some possible answers are:

- The time delay cannot be negated by the compensation link, even if the phase margin and cross-over frequency are recovered. The response to a reference change or a disturbance will necessarily be delayed by 32 ms, no matter what controller is used.

- The lead link will decrease the low-frequency gain of the system (asymptotic gain of 0.5), which means that static disturbances will be eliminated more slowly.

- The lead link will increase the high-frequency gain of the system (asymptotic gain of 2), which means that the system will be more sensitive to measurement noise.