LUNDS UNIVERSITET

Institutionen för
REGLERTEKNIK

Automatic Control, Basic Course FRTF05

Exam 08 April 2021, 08:00–13:00

Points and grades

All solutions must be well motivated. The exam total is 25 points. The number of points are presented after each problem.

Preliminary grades:

Grade 3: at least 12 points
4: at least 17 points
5: at least 22 points

Allowed aids

All course material, other material, and computer resources are allowed (including lecture notes, exercise manual, Matlab, ...) but no collaboration or communication.

Results

Exam results are communicated via LADOK.

1. A process is represented by the differential equation

$$2\ddot{y}(t) + b\dot{y}(t) + 8y(t) = 0.2\dot{u}(t) + 10u(t).$$

a. For which values of b is the system asymptotically stable? (1 p)

b. For which values of $b > 0$ does the system have complex poles? (1 p)

c. Let $b = 8$. Sketch the Bode diagram of the system (both amplitude and phase diagram). (2 p)

Solution

We start by dividing by 2 in order to get a monic characteristic polynomial. Laplace transform then gives:

$$s^2 Y(s) + s\frac{b}{2}Y(s) + 4Y(s) = s\frac{1}{10}U(s) + 5U(s).$$

This gives the following transfer-function:

$$\frac{Y(s)}{U(s)} = \frac{0.1s + 5}{s^2 + 0.5bs + 4}.$$

a. A system with characteristic polynomial $s^2 + a_1 s + a_2$ is asymptotically stable if $a_1 > 0$ and $a_2 > 0$. Thus the system is asymptotically stable $\forall b > 0$.

b. Solving $s^2 + 0.5bs + 4 = 0$ we get

$$s = -\frac{b}{4} \pm \sqrt{-4 + \frac{b^2}{16}}$$

Here it can be seen that imaginary poles are given when $-4 + \frac{b^2}{16} < 0$. Thus $|b| < 8$ results in complex-valued poles.

c. For $b = 8$ the characteristic polynomial can be factorized as:

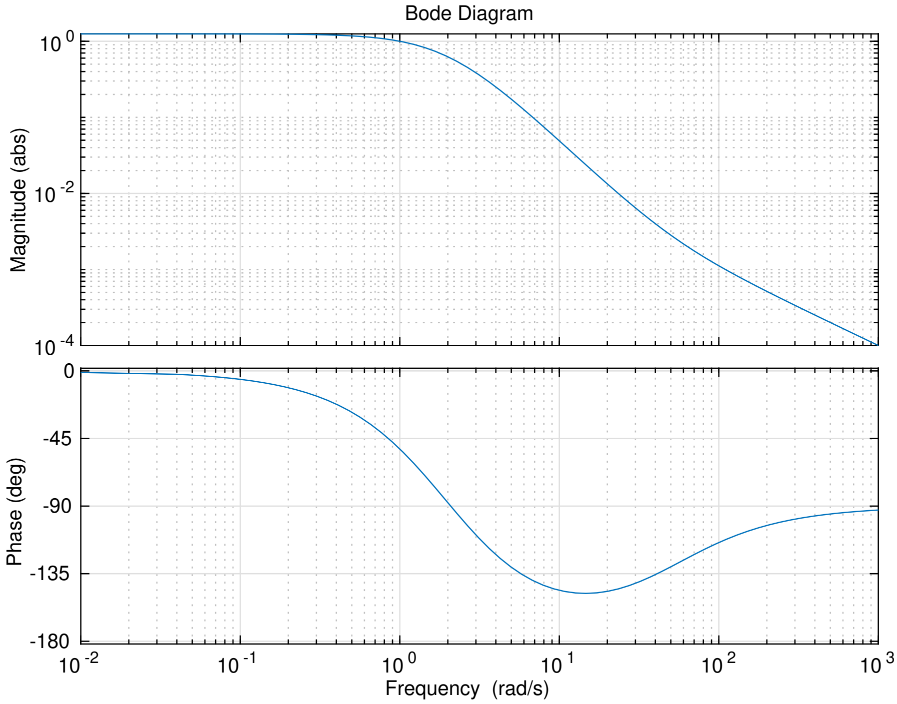
$$s^2 + 4s + 4 = (s + 2)^2$$

Giving the following transfer function:

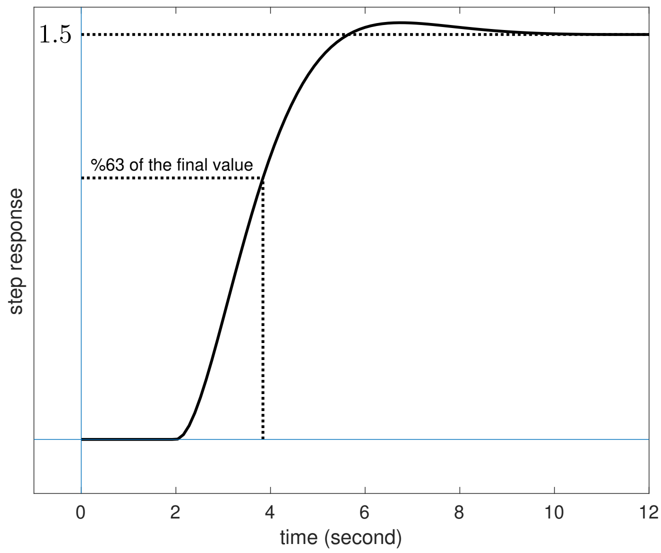
$$G(s) = \frac{Y(s)}{U(s)} = 0.1\frac{s + 50}{(s + 2)^2}$$

Thus, for the Bode magnitude diagram we will have a slope of 0 for $\omega < 2$, a slope of -2 for $2 < \omega < 50$ and a slope of -1 for $\omega > 50$. The low frequency gain is given by $G(0) = \frac{5}{4} = 1.25$. The Bode phase diagram will start at $0°$ and hit $-90°$ for $\omega = 2$. The phase will tend towards $-180°$ but will not make it all the way as the zero takes it back up to towards $-90°$ for large ω. See Fig. 1.

2. The step response (after a unit step) for a system is shown in Figure 2. Tune a PID-controller using the Lambda method with $\lambda = T$, whre T is the estimated time constant. (2 p)



Figur 1 Bode diagram for Problem **1c**.



Figur 2 Step response of the system in Problem **2**.

Solution By drawing the tangent to the inflection point of the step response, an approximation of the dead time is obtained, $L \simeq 2.35$. The step response has reached 63 percent of its final value after approximately 3.85 seconds. The time constant thus becomes $T \simeq 3.85 - 2.35 = 1.5$. The static gain $K_p = 1.5$. With

$\lambda = T$ the PID controller parameters become

$$K = 0.66 \qquad T_i = 2.67 \qquad T_d = 0.66$$

3. Match the following transfer functions with their Nyquist plot in Figure 3. Motivate your answers. (2 p)

$$G_1(s) = \frac{s+4}{(s+2)^2} \qquad G_2(s) = \frac{2e^{-s}}{s+1}$$
$$G_3(s) = \frac{4}{(s+1)(s+2)^2} \quad G_4(s) = \frac{2}{s(s+2)}$$

Solution

G_1: The phase has to start at 0° and end at −90°, as at very large frequencies, we get −180° contribution from the denominator and +90° contribution from the numerator. So, G_1 matches with F.
G_2: Because of the delay, we should see a spiral curve in the Nyquist plot. Since we don't have an integrator in the transfer function, the starting angle should be 0°. Moreover, at zero frequency the magnitude has to be 2. So G_2 corresponds to A.
G_3: It is a third order system (without integrator), so the phase should start at 0° and end at −270° (tangent to imaginary axis). G_3 matches with B.
G_4: Because of the integrator in the transfer function, the curve should start at −90° and we don't have delay in the transfer function, so we don't expect a spiral in the Nyquist plot. Hence, G_4 corresponds to C.

4. Consider the differential equation

$$\ddot{y} + (y+1)\dot{y} + y^2 = u.$$

a. Introduce the state variables $x_1 = y$ and $x_2 = \dot{y}$ and write the system dynamics in state space form. (1 p)

b. Find all stationary points, (x_1^*, x_2^*, u^*), for the system. (1 p)

c. Linearize the system around the stationary point where $x_1^* = 2$. (2 p)

Solution

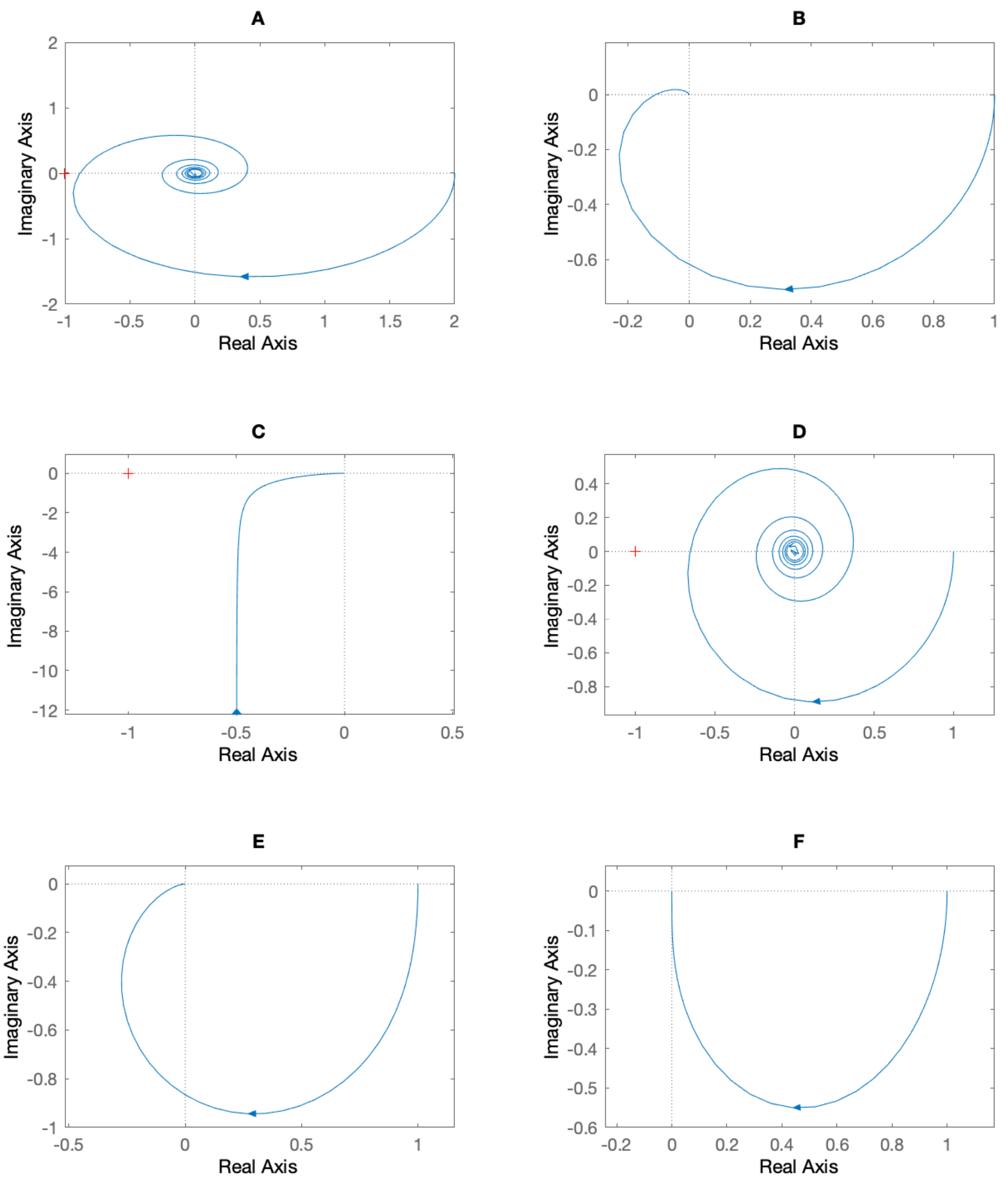
a. Introducing $x_1 = y$ and $x_2 = \dot{y}$, we get the following state-space representation:

$$\begin{aligned}\dot{x}_1 &= x_2 := f_1(x_1, x_2, u)\\ \dot{x}_2 &= -(x_1+1)x_2 - x_1^2 + u := f_2(x_1, x_2, u)\\ y &= x_1 := g(x_1, x_2, u)\end{aligned}$$

b. All points where $\dot{x}_1 = 0$, $\dot{x}_2 = 0$ and $\dot{u} = 0$ are stationary points. Inserting into the state space equations yields

$$\begin{cases} 0 = x_2^* \\ 0 = -(x_1^*+1)x_2^* - (x_1^*)^2 + u^* \end{cases} \Leftrightarrow \begin{cases} x_2^* = 0 \\ (x_1^*)^2 = u^* \end{cases} \Leftrightarrow \begin{cases} x_2^* = 0 \\ x_1^* = \pm\sqrt{u^*} \end{cases}$$

Thus, x_2^* has to be zero for any stationary point and $x_1^* = \pm\sqrt{u*}$, so u^* must be nonnegative.



Figur 3 The Nyquist plots for Problem **3**.

c. Setting $x_1^* = 2$ implies $u^* = 4$ and introducing $x = (x_1, x_2)$ and $f = (f_1, f_2)$, Taylor expansion yields

$$\dot{x} = f(x,u) \approx f(x^*,u^*) + \frac{\partial f(x^*,u^*)}{\partial x}(x - x^*) + \frac{\partial f(x^*,u^*)}{\partial u}(u - u^*)$$
$$y = g(x,u) \approx g(x^*,u^*) + \frac{\partial g(x^*,u^*)}{\partial x}(x - x^*) + \frac{\partial g(x^*,u^*)}{\partial u}(u - u^*)$$

where

$$f(x^*, u^*) = \begin{bmatrix} f_1(x^*, u^*) \\ f_2(x^*, u^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad g(x^*, u^*) = x_1^* = y^* = 2$$

and

$$\begin{aligned}
\frac{\partial f(x^*, u^*)}{\partial x} &= \begin{bmatrix} \frac{\partial f_1(x^*,u^*)}{\partial x_1} & \frac{\partial f_1(x^*,u^*)}{\partial x_2 n} \\ \frac{\partial f_2(x^*,u^*)}{\partial x_1} & \frac{\partial f_2(x^*,u^*)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -x_2^* - 2x_1^* & -x_1^* - 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \\
\frac{\partial f(x^*, u^*)}{\partial u} &= \begin{bmatrix} \frac{\partial f_1(x^*,u^*)}{\partial u} \\ \frac{\partial f_2(x^*,u^*)}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\frac{\partial g(x^*, u^*)}{\partial x} &= \begin{bmatrix} \frac{\partial g(x^*,u^*)}{\partial x_1} & \frac{\partial g(x^*,u^*)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \\
\frac{\partial g(x^*, u^*)}{\partial u} &= 0
\end{aligned}$$

Introducing also $\Delta x = x - x^*$, $\Delta y = y - y^*$ and $\Delta u = u - u^*$, we get $\dot{\Delta x} = \dot{x}$. The linearized state space equations can be written as:

$$\begin{aligned}
\dot{\Delta x} &= \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u \\
\Delta y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \Delta x.
\end{aligned}$$

5. We consider the following system

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x.
\end{aligned}$$

a. Assume that the system is initiated at $x(0) = (3, 7)^T$ and that the control signal is the zero constant, $u(t) = 0$. Will $x(t) \to 0$ as $t \to \infty$? Motivate your answer. (1 p)

b. Assume that the system is initiated at $x(0) = (0, 0)^T$. Is it possible to choose control signal $u(t)$ (not necessarily constant) so that $x(\tau) = (3, 7)^T$ at some finite point in time $\tau > 0$? Motivate your answer. (1 p)

c. Suppose that we can measure all both states x_1 and x_2 and that we want to control the system using $u(t) = -l_1 x_1 - l_2 x_2 + l_r r$ for some constants l_1, l_2, and l_r. Decide l_1, l_2 and l_r so that the closed loop system, that is the system from reference r to measurement y, has two poles in $s = -3$ and static gain of 1. (3 p)

Solution

a. **Yes.** The eigenvalues for A are $\lambda = -2, -3$. This means that that the poles are $s = -2, -3$. Therefore, the system is **asymptotically stable**. This implies that for an arbitrary initial $x(0)$ we will have $x(t) \to 0$ as $t \to \infty$ if $u(t) = 0$.

b. **Yes.** That a system is **controllable** means that if it is initiated $x(0)$ a control signal $u(t)$ can be chosen such that an arbitray state $x(\tau)$ can be reached at a finite time τ. The controllability matrix is

$$W_s = [\,B \quad AB\,] = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}.$$

It has two linearly independent columns, which implies that W_s has full rank and that the system is controllable.

c. The system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u = Ax + Bu \\ y &= [\,1 \quad 0\,]\, x = Cx \end{aligned} \tag{1}$$

We want to design a state feedback so that $u(t) = l_r r(t) - Lx(t)$, with $L = [\,l_1 \quad l_2\,]$, so that the closed loop system (from r to y) has characteristic polynomial

$$(s+3)^2 = s^2 + 6s + 9. \tag{2}$$

Inserting the control law into (1) gives closed loop system

$$\begin{aligned} \dot{x} &= (A - BL)x + Bl_r r \\ y &= Cx. \end{aligned}$$

The poles are the eigenvalues to $A - BL$, which are given by zeros to the characteristic polynomial

$$\begin{aligned} p(s) = \det(sI - (A - BL)) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} [\,l_1 \quad l_2\,] \right) \\ &= \det \begin{bmatrix} s+2 & -1 \\ 2l_1 & s+3+2l_2 \end{bmatrix} = s^2 + (5 + 2l_2)s + 6 + 4l_2 + 2l_1. \end{aligned}$$

By matching with (2), we get

$$\begin{cases} 5 + 2l_2 = 6 \\ 6 + 2l_1 + 4l_2 = 9 \end{cases} \quad \Leftrightarrow \quad \begin{cases} l_1 = 1/2 \\ l_2 = 1/2 \end{cases}.$$

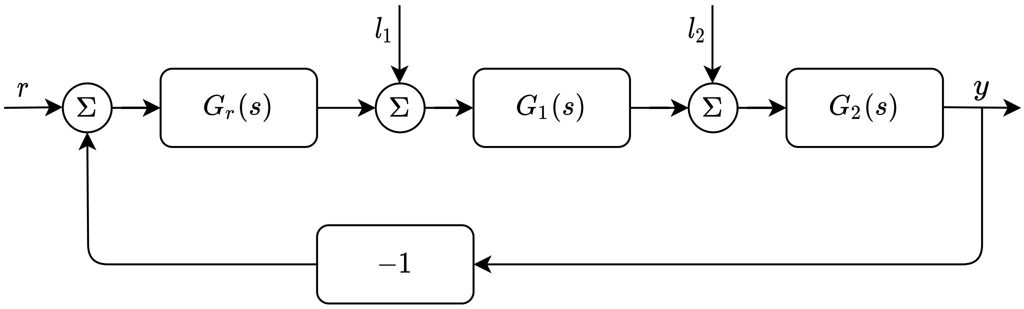
The static gain is given by $G(0)$. The transfer function for the closed loop system is

$$G(s) = C(sI - (A - BL))^{-1} B l_r = [\,1 \quad 0\,] \begin{bmatrix} s+2 & -1 \\ 1 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} l_r.$$

We get

$$G(0) = [\,1 \quad 0\,] \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} l_r = [\,1 \quad 0\,] \frac{1}{9} \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} l_r = \frac{2l_r}{9},$$

and the requirement that the static gain should be 1, $G(0) = 1$, is satisfied if $l_r = 9/2$.



Figur 4 Block diagram for Problem **6**.

6. A process $G_p(s)$ consists of two components, so that $G_p(s) = G_2(s)G_1(s)$. We want to control this process with one controller $G_r(s)$. Load disturbances can occur either before the first component $G_1(s)$ or between the two components $G_1(s)$ and $G_2(s)$. The block diagram for the closed loop system is shown in Figure 4. Suppose that we select a PD-controller with $T_d = 1/K$ and a K that satisfies $0 < K < 1$ and that the system components then are

$$G_r(s) = K + s, \quad G_1(s) = \frac{1}{s(s+1)}, \quad G_2(s) = \frac{1}{s}.$$

We also suppose that the reference signal $r(t) = 0$ for all times t.

a. Find the transfer function from load disturbance l_1 to measurement y. (1 p)

b. Find the transfer function from load disturbance l_2 to measurement y. (1 p)

c. What is $y(t)$ as $t \to \infty$ if $l_1(t)$ is a unit step and $l_2(t) = 0$? (1 p)

d. What is $y(t)$ as $t \to \infty$ if $l_1(t) = 0$ and $l_2(t) = t$? (1 p)

e. What is $y(t)$ as $t \to \infty$ if $l_1 = \sin t$ and $l_2 = 0$? (Remember that $0 < K < 1$.) (2 p)

Solution

a. We can set the other external signals to zero, i.e., $r = 0$ and $l_2 = 0$. We get

$$Y(s) = G_2(s)G_1(s)[L_1(s) + G_r(s)(-Y(s))] \Rightarrow Y(s) = \frac{G_2(s)G_1(s)}{1 + G_2(s)G_1(s)G_r(s)} L_1(s),$$

and the transfer function is

$$\frac{Y(s)}{L_1(s)} = \frac{G_2(s)G_1(s)}{1 + G_2(s)G_1(s)G_r(s)} = \frac{1/(s^2(s+1))}{1 + (K+s)/(s^2(s+1))} = \frac{1}{s^3 + s^2 + s + K}.$$

b. We can set $r = 0$ och $l_1 = 0$. We get

$$Y(s) = G_2(s)[L_2(s) + G_1(s)G_r(s)(-Y(s))] \Rightarrow Y(s) = \frac{G_2(s)}{1 + G_2(s)G_1(s)G_r(s)} L_2(s),$$

and the transfer function is

$$\frac{Y(s)}{L_2(s)} = \frac{G_2(s)}{1 + G_2(s)G_1(s)G_r(s)} = \frac{1/s}{1 + (K+s)/(s^2(s+1))} = \frac{s(s+1)}{s^3 + s^2 + s + K}.$$

c. That l_1 is a unit step implies that $L_1(s) = \frac{1}{s}$. The final value theorem gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{G_2(s)G_1(s)}{1+G_2(s)G_1(s)G_r(s)}\frac{1}{s} = \lim_{s\to 0}\frac{1}{s^3+s^2+s+K} = \frac{1}{K}.$$

This can be applied since the pole polynomial s^3+s^2+s+K has positive coefficients and satisfies $a_1a_2 = 1\cdot 1 > a_3 = K$.

d. That $l_2(t) = t$ implies that $L_2(s) = \frac{1}{s^2}$. The final value theorem gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{G_2(s)}{1+G_2(s)G_1(s)G_r(s)}\frac{1}{s^2} = \lim_{s\to 0}\frac{s+1}{s^3+s^2+s+K} = \frac{1}{K}.$$

This can be applied since the pole polynomial s^3+s^2+s+K has positive coefficients and satisfies $a_1a_2 = 1\cdot 1 > a_3 = K$.

e. That $l_1(t) = \sin t$ implies that $L_1(s) = \frac{1}{s^2+1}$. We get

$$sY(s) = s\frac{G_2(s)G_1(s)}{1+G_2(s)G_1(s)G_r(s)}\frac{1}{s^2+1} = \frac{s}{(s^3+s^2+s+K)(s^2+1)}.$$

The factor (s^2+1) in the pole polynomial implies two poles on the imaginary axis. The system is not asymptotically stable and the final value theorem cannot be applied.

We do know that a linear system with sinusoidal input always has a sinusoidal output with the same frequency ω, but amplitude multiplied by $|G(i\omega)|$ and phase shift given by $\arg G(i\omega)$. In this case, we get

$$G(i\omega) = \frac{1}{(i\omega)^3+(i\omega^2)+i\omega+K} = \frac{1}{K-\omega^2+i(\omega-\omega^3)}.$$

Inserting $\omega = 1$ gives $G(i1) = 1/(K-1)$, and the output is

$$y(t) = |G(i1)|\sin(t+\arg G(i1)) = \frac{1}{|K-1|}\sin(t-\pi) = \frac{1}{K-1}\sin t,$$

since $K < 1$.

7. Consider the following transfer function

$$G(s) = \frac{Ke^{-s}}{s(s+2)}.$$

Assume that we close the loop with the feedback gain of -1 (which gives a closed loop transfer function of $G_{cl}(s) = \frac{G(s)}{1+G(s)}$). Following the steps below, find an upper bound $\bar{K} > 0$ for which all K satisfying $0 < K < \bar{K}$ gives a stable closed loop system.

a. Find the expressions that describes the phase and magnitude of $G(i\omega)$, that is $\arg G(i\omega)$ and $|G(i\omega)|$. (0.5 p)

b. Find ω_0 such that $\arg G(i\omega_0) = -\pi$. (Hint: You can use Matlab to solve the resulting non-linear algebraic equation). (0.5 p)

c. Find K such that $|G(i\omega_0) = 1$. (0.5 p)

d. Based on **c**, find an upper bound $\bar{K} > 0$ for which all K satisfying $0 < K < \bar{K}$ gives a stable closed loop system. (0.5 p)

Solution

a. The expression that describes the phase of $G(i\omega)$ is

$$\arg G(i\omega) = \frac{-\pi}{2} - \tan^{-1}(\frac{\omega}{2}) - \omega$$

and the expression that describes the magnitude of $G(i\omega)$ is

$$|G(i\omega)| = \frac{K}{\omega\sqrt{4+\omega^2}}$$

b.

$$\begin{aligned}\arg G(i\omega_0) &= \frac{-\pi}{2} - \tan^{-1}(\frac{\omega_0}{2}) - \omega_0 = -\pi \\ \Rightarrow \qquad \frac{\omega_0}{2} &= \tan(\pi/2 - \omega_0) \qquad \Rightarrow \qquad \omega_0 = 1.077\text{rad/s}\end{aligned}$$

c.

$$|G(i\omega_0)| = \frac{K}{\omega_0\sqrt{4+\omega_0^2}} = 0.4088K = 1 \qquad \Rightarrow \qquad K = 2.4465$$

d. The Nyquist curve cuts (-1,0) for the first time at $K = 2.4465$. Since no poles are in the right half plane and the pole in the origin is unique, the Nyquist criterion implies that $\bar{K} = 2.4465$.