LUNDS UNIVERSITET

Institutionen för
REGLERTEKNIK

Automatic Control, Basic Course FRTF05

Exam March 17 2021, 08:00–13:00

Points and grades

All solutions must be well motivated. The exam total is 25 points. The number of points are presented after each problem.

Preliminary grades:

Grade 3: at least 12 points
4: at least 17 points
5: at least 22 points

Allowed aids

All course material, other material, and computer resources are allowed (including lecture notes, exercise manual, Matlab, ...) but no collaboration or communication.

Results

Exam results are communicated via LADOK.

1. A system is described by the following differential equation

$$\ddot{y} + \dot{y} + \sin y = u.$$

a. Introduce the state variables $x_1 = y$ and $x_2 = \dot{y}$, and write down the corresponding state-space form for the system. (1 p)

b. Determine all stationary points (x_1^0, x_2^0, u^0) for the system for which $u^0 = 1$. (1 p)

c. Linearize the system around the stationary point which has the smallest positive value for x_1^0. (1 p)

Solution

a. With the state variables $x_1 = y$ and $x_2 = \dot{y}$, we get the state-space form

$$\begin{aligned} \dot{x}_1 &= x_2 =: f_1(x_1, x_2, u) \\ \dot{x}_2 &= -x_2 - \sin x_1 + u =: f_2(x_1, x_2, u) \\ y &= x_1 =: g(x_1, x_2, u). \end{aligned}$$

b. Stationary points are the points for which $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, i.e. the points that fulfil

$$\begin{cases} 0 = x_2 \\ 0 = -x_2 - \sin x_1 + u. \end{cases}$$

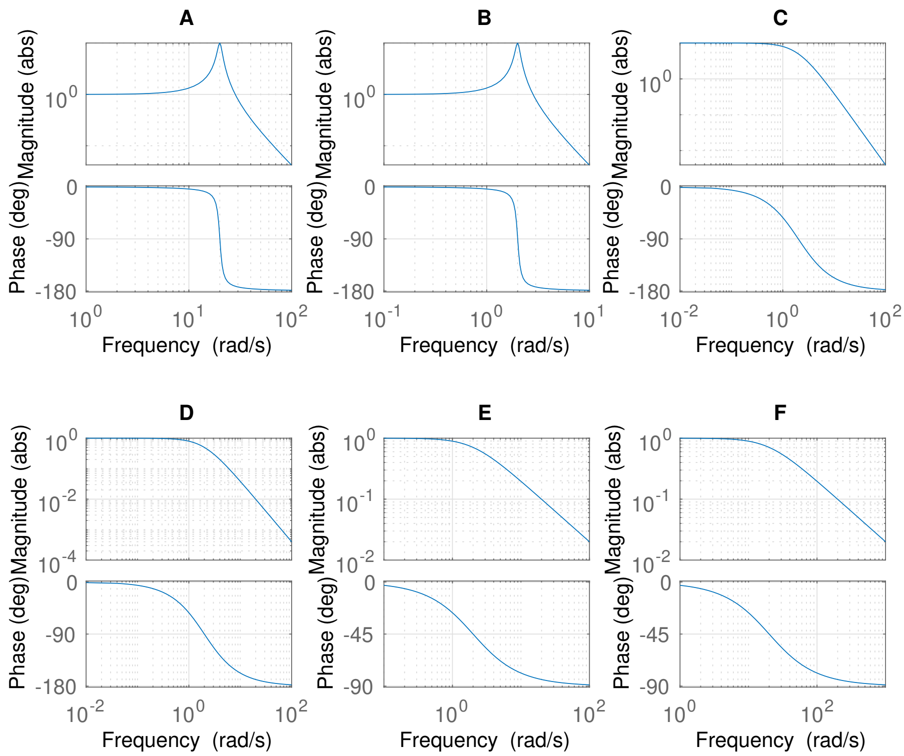
All such points with $u^0 = 1$ are given by $(x_1^0, x_2^0, u^0) = (\frac{\pi}{2} + 2\pi n, 0, 1)$, for $n = 0, \pm 1, \pm 2, \ldots$.

c. The stationary point with smallest positive value of x_1^0 is $(x_1^0, x_2^0, u^0) = (\frac{\pi}{2}, 0, 1)$. The partial derivatives for the system equations are

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial u} = 0, \\ &\frac{\partial f_2}{\partial x_1} = -\cos x_1, \quad \frac{\partial f_2}{\partial x_2} = -1, \quad \frac{\partial f_2}{\partial u} = 1, \\ &\frac{\partial g}{\partial x_1} = 1, \quad \frac{\partial g}{\partial x_2} = 0, \quad \frac{\partial g}{\partial u} = 0. \end{aligned}$$

With $x = [\, x_1 \quad x_2 \,]^T$, we introduce the variables $\Delta x = x - x^0$, $\Delta u = u - u^0$ and $\Delta y = y - y^0$, giving

$$\begin{aligned} \frac{\mathrm{d}\Delta x}{\mathrm{d}t} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{(x_1^0, x_2^0, u^0)} \Delta x + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\Bigg|_{(x_1^0, x_2^0, u^0)} \Delta u \\ &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u \\ \Delta y &= [\, \tfrac{\partial g}{\partial x_1} \quad \tfrac{\partial g}{\partial x_2} \,]\big|_{(x_1^0, x_2^0, u^0)} \Delta x + \frac{\partial g}{\partial u}\bigg|_{(x_1^0, x_2^0, u^0)} \Delta u = [\, 1 \quad 0 \,]\, \Delta x. \end{aligned}$$



Figur 1 Bode diagrams for problem 2.

2. Consider the following three systems:

$$G_\alpha(s) = \frac{20}{s+20}, \quad G_\beta(s) = \frac{4}{(s+2)^2},$$
$$G_\gamma(s) = \frac{1}{(1+s/(1-20i))(1+s/(1+20i))}.$$

a. State which of the Bode diagrams A-F in Figure 1 that belong to each of the three transfer functions. Motivate your answers. (1.5 p)

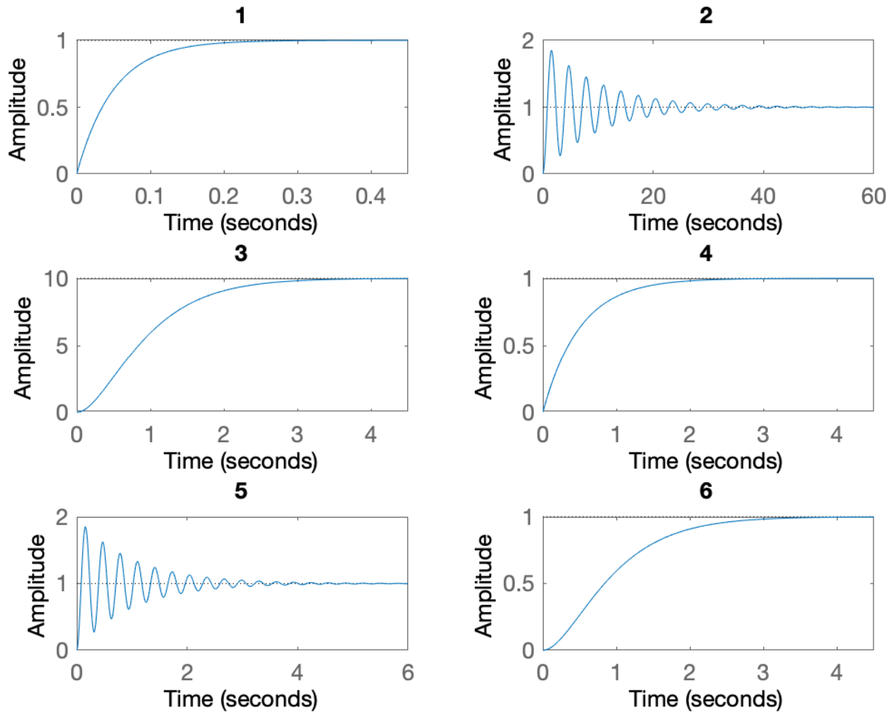
b. State which of the three step responses 1-6 in Figure 2 that belong to each of the three transfer functions. Motivate your answers. (1.5 p)

c. State which of the three transfer functions that is described by the Nyquist diagram in Figure 3. Motivate your answer. (1 p)

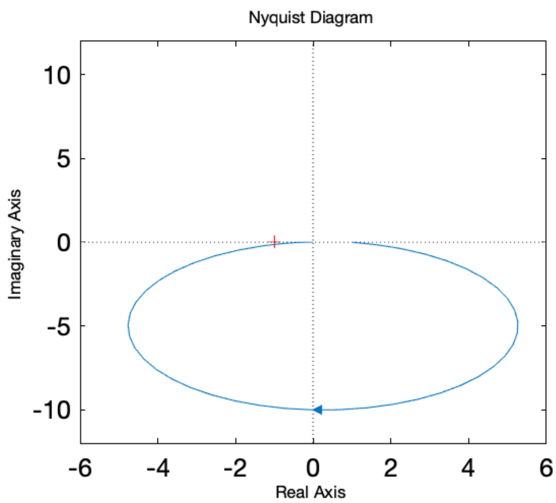
Solution

a. $G_\alpha(s)$ is a first-order system, which means that the phase curve cannot go lower than to $-90°$. The system must therefore correspond to E or F. The pole in $s = -20$ gives a break frequency $\omega = 20$ in the Bode diagram, which we have for diagram F. Hence we have $G_\alpha(s) \leftrightarrow$ **F**.

$G_\beta(s)$ is a second-order system with real poles. Since we have two poles and no zeros, the phase curve must go down to $-180°$. The diagrams E and F can thus be excluded. In A and B, distinct resonance peaks can be seen, which



Figur 2 Step responses for problem 2.



Figur 3 Nyquist diagram for problem 2.

requires complex poles with a large imaginary part. These can therefore also be excluded. The static gain is given by $G(0)$, and we see that $G_\beta(0) = 1$. The one of the diagrams C and D that has static gain (gain at low frequencies) 1 is D, so we therefore have $G_\beta(s) \leftrightarrow$ **D**.

$G_\gamma(s)$ is a second-order system with two complex conjugated poles with large imaginary part. A large imaginary part relative to the real part means that the system is resonant, and this scan be seen as a resonance peak in the Bode diagram. The system must thus correspond to A or B. When the relative damping ζ for the complex conjugated pole pair is close to zero, the resonance frequency of the system is approximately gives as the distance of the poles from the origin: $\omega_0 = \sqrt{(-1)^2 + 20^2} \approx 20$. Thus, the system has a resonance frequency of about 20 rad/s, which is the position of the resonance peak in diagram A. We therefore have $G_\gamma(s) \leftrightarrow$ **A**.

b. The system $G_\alpha(s)$ is a first-order system, and therefore does not have any complex poles. Since it neither has any zeros, we cannot get any resonances in the step response, and step responses 2 and 5 can thus be excluded. For a first-order system, the derivative for a step response at $t = 0$ is non-zero. This can be shown by the initial value theorem. Therefore, 3 and 6 can be excluded. Alternatively, we could look up the function for the step response in the collection of formulae, as the inverse Laplace transform of $Y(s) = \frac{20}{s+20}\frac{1}{s}$, which is $y(t) = 1 - e^{-20t}$, and note that the derivative $\dot{y}(t) = 20e^{-20t}$ is strictly decreasing, which also excludes 3 and 6. The time constant for the system is $T = 1/20$. This is the time it takes for the step response to reach approximately 63% of its final value (which follows from $y(T) = 1 - e^{-20T} = 1 - e^{-1} \approx 0.63$). This should therefore take about 0.05 seconds, and we then see that $G_\alpha(s) \leftrightarrow$ **1**.

$G_\beta(s)$ is a second-order system with real poles. Since it has two poles and no zeros, the initial value theorem gives that the initial derivative is 0. We can then exclude 1 and 4. Since we have real poles only and no zeros, we cannot get an overshoot, so 2 and 5 can therefore be excluded. The static gain is $G_\beta(0) = 1$, so the stationary value of the step response should be 1. Thus we must have $G_\beta(s) \leftrightarrow$ **6**.

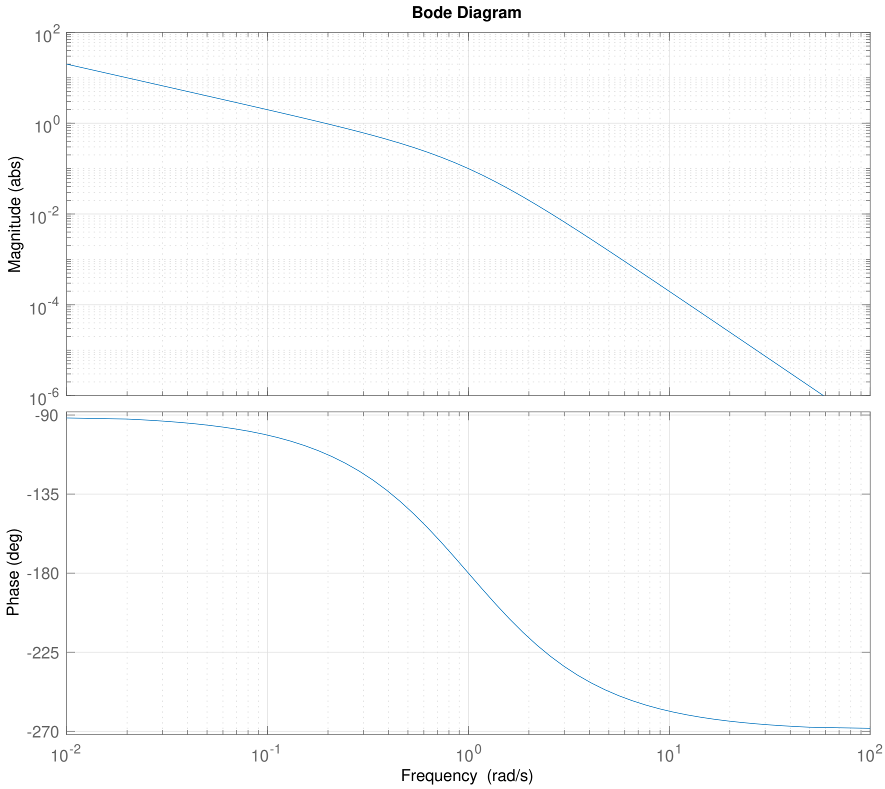
Since $G_\gamma(s)$ is a strongly resonant system, we should see resonances in the step response. So it cannot be 2 or 5. The frequency of the resonances in the step response should correspond to the resonance frequency of the system. We see in 5 that the step response resonates with about 6.5 periods in 2 seconds, giving a frequetncy $f = 6.5/2 = 3.25$ Hz. The angular frequency is then $\omega = 2\pi f \approx 20$ rad/s. An analogue calculation for 2 gives an angular frequency that is about a tenth as large. So we must have $G_\gamma(s) \leftrightarrow$ **5**.

c. The only one of the transfer functions that has a gain $|G(i\omega)| > 1$ for any ω is $G_\gamma(s)$, because of the resonance peak. Since the distance to the origin from some of the points on the Nyquist curve is larger than 1, the Nyquist curve must hence correspond to the system $G_\gamma(s)$.

3. A closed-loop system (simple feedback) has the open-loop transfer function $G_0(s)$, without poles in the right half plane, for which the Bode diagram is shown in Figure 4. State whether each of the statements below is true or false, and motivate your answers.

a. The open-loop system $G_0(s)$ has a pole in $s = 0$. (0.5 p)

b. The open-loop system $G_0(s)$ contains an integrator. (0.5 p)



Figur 4 Bode diagram for open-loop system G_0 in Problem 3.

c. The open-loop system $G_0(s)$ contains a delay. (0.5 p)

d. The open-loop system has at least three poles. (0.5 p)

e. If $G_0(s)$ is replaced by $5G_0(s)$, then the closed-loop system becomes unstable. (0.5 p)

f. If we in $G_0(s)$ add a delay of 3 seconds, then the closed-loop system becomes unstable. (0.5 p)

Solution

a. True. We see that the gain curve goes to infinity when $\omega \to 0$, or equivalently that the low-frequency asymptote has a negative slope. This means that we have a pole in $s = 0$, since a factor $1/s$ in the transfer function gives a factor $1/\omega$ in the gain $|G(i\omega)|$, which is required for the gain to go to infinity when $\omega \to 0$.

b. True. A pole in $s = 0$ is equivalent to that the system contains an integrator.

c. False. A delay gives a phase curve that goes to $-\infty$ for large frequencies, but the phase curve does not go lower than to $-270°$.

d. True. Each pole can decrease the phase with at most 90°. Since the phase decreases to −270°, we must have at least three poles (since the system does not have any delay, which would be the other way to decrease the phase). Alternatively we can see that the gain curve has the slope −3 for large frequencies, which requires at least three poles, even if the system would have a delay.

e. False. We can compute the amplitude margin of the system as $A_m = 1/|G_0(i\omega_0)| \approx 1/0.1 = 10$, where ω_0 is the frequency for which $\arg G_0(i\omega_0) = -180°$. Since the amplitude margin is 10, we can multiply $G_0(s)$ with a positive constant that is less than 10 without making the system unstable.

f. False. The delay margin for the system is given by $L_m = \varphi_m/\omega_c$, where ω_c is the cut-off frequency, i.e. the frequency for which $|G_0(i\omega_c)| = 1$, and $\varphi_m = \pi - |\arg G_0(i\omega_c)|$ is the phase margin. We see that $\omega_c \approx 0.2$ and the phase margin is approximately $\varphi_m = (3/4)\cdot(\pi/2) = 3\pi/8$. We then get $L_m \approx (3\pi/8)/0.2 \approx 5.9$, so the delay margin is about 6 seconds, which means that we can add a delay of 3 seconds without making the system unstable.

4. Consider the block diagram of Figure 5.

a. Find the transfer functions from d to u and from n to u. (1 p)

b. We can describe the output y as

$$y = G_{y/r}r + G_{y/d}d + G_{y/n}n.$$

Find the transfer functions $G_{y/r}$, $G_{y/d}$, and $G_{y/n}$. (1.5 p)

c. Let $d = 0$, $n = 0$, and

$$P(s) = \frac{1}{s^2 + 2s - 2} \qquad C(s) = K(1 + \frac{1}{T_i s}).$$

Find valid ranges of K and T_i such that the closed loop system becomes stable. (2 p)

d. Let $K = 4$, $T_i = 10$, and $H = 0$. What is the stationary value of the output y when all the inputs are unit steps, i.e. $r(t) = d(t) = n(t) = 1$ for all $t \geq 0$. (1.5 p)

e. Again let $K = 4$, $T_i = 10$, and $H = 0$. What is the stationary value of the output y when all the inputs are unit ramps, i.e. $r(t) = d(t) = n(t) = t$ for all $t \geq 0$. (1.5 p)

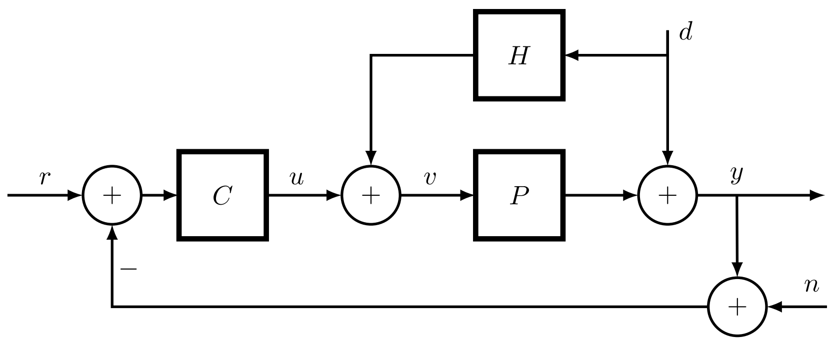
Solution

a. For the transfer function from d to u we have

$$-C(P(Hd + u) + d) = u \quad \Rightarrow \quad -CPHd - Cd = CPu + u$$
$$\Rightarrow \quad u = \frac{-C(PH + 1)}{1 + CP}d$$

and for the transfer function from n to u we get

$$-C(Pu + n) = u \quad \Rightarrow \quad u = \frac{-C}{1 + CP}n$$



Figur 5 The control loop in Problem 4.

b.

$$P(C(r-n-y)+Hd)+d=y \quad \Rightarrow \quad y=\frac{PC}{1+PC}r+\frac{1+PH}{1+PC}d+\frac{-PC}{1+PC}n$$

c. The characteristic polynomial is

$$T_i s(s^2+2s-2)+KT_i s+K=T_i s^3+2T_i s^2+T_i(K-2)s+K=0$$

reformulating the characteristic polynomial gives

$$s^3+2s^2+(K-2)s+K/T_i=0 \quad \Rightarrow$$
$$K-2>0, \qquad K/T_i>0, \qquad 2(K-2)>K/T_i$$

which give

$$K>2, \quad T_i>\frac{K}{2(K-2)}$$

d. To find the stationary value, we can use the final value theorem (the conditions are satisfied)

$$\begin{aligned} y(\infty) &= \lim_{s\to 0} sY(s)=\lim_{s\to 0} s(\frac{PC}{1+PC}+\frac{1+PH}{1+PC}+\frac{-PC}{1+PC})\frac{1}{s} \\ &= \lim_{s\to 0}(\frac{PC+1-PC}{1+PC})=\lim_{s\to 0}(\frac{1}{1+PC})=\lim_{s\to 0}\frac{T_i s(s^2+2s-2)}{T_i s(s^2+2s-2)+K(1+T_i s)}=0 \end{aligned}$$

e. To find the stationary value, we can use the final value theorem (the conditions are satisfied)

$$\begin{aligned} y(\infty) &= \lim_{s\to 0} sY(s)=\lim_{s\to 0} s(\frac{PC}{1+PC}+\frac{1+PH}{1+PC}+\frac{-PC}{1+PC})\frac{1}{s^2} \\ &= \lim_{s\to 0}(\frac{PC+1-PC}{1+PC})\frac{1}{s}=\lim_{s\to 0}(\frac{1}{1+PC})\frac{1}{s} \\ &= \lim_{s\to 0}\frac{T_i s(s^2+2s-2)}{T_i s(s^2+2s-2)+K(1+T_i s)}\frac{1}{s}=\frac{-2T_i}{K}=-5 \end{aligned}$$

5. Kim wants to control the system

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

with a state-feedback controller, and has designed the following control law

$$u = -Lx + l_r r,$$

where $l_r = 13$ and

$$L = \begin{bmatrix} 17.25 & 7 \end{bmatrix}$$

Afterwards, Kim realizes that all states are not measurable and thinks that the control design has failed. Luckily you are there to tell him that it might be possible to use the control law anyway, this by designing an observer.

a. Show that the system is observable. (1 p)

b. Help Kim by designing an observer suitable for the system and the proposed controller (motivate your design choices). (3 p)

Solution

a. To find out whether the system is observable or not, we construct the obeservability and matrix check for linear dependency in it's columns.

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

As W_o is a 2x2 matrix and

$$\text{rank}(W_o) = 2$$

we have full rank and thus linearly independent columns, and the system is observable. Another way to verify this would be to ensure that $\det(W_o) \neq 0$.

b. A rule of thumb is to place the poles of the observer at the same angle as the closed loop system poles but at least 2 times further away from the origin.

We start by finding the poles of the closed loop system with the given control law. For a system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with the control law

$$u = -Lx + l_r r$$

the closed loop transfer-function is given by

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - (A - BL))^{-1} B l_r r.$$

The poles are found by solving

$$\det(sI - (A - BL)) = 0.$$

For the given values, we have

$$0 = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 17.25 & 7 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s-1 & -2 \\ 32.5 & s+13 \end{bmatrix}\right)$$
$$= (s-1)(s+13) - (-65)$$
$$= s^2 + 12s - 13 + 65.$$

Solving for s yields:

$$s = -6 \pm 4i.$$

The observer should be designed such that the poles are at least 2 times faster than the poles of the closed loop system with state feedback. We select exactly twice as fast and place the observer poles at $p_{1,2} = -12 \pm 8i$.

The observer is of the form

$$\dot{\hat{x}} = A\hat{x} + Bu + K(Cx - C\hat{x}).$$

The dynamics for the error between the real states and the estimated states is

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - K(Cx - C\hat{x})$$
$$= (A - KC)\tilde{x}.$$

We will now find K such that $(A - KC)$ gets the desired eigenvalues, $p_{1,2} = -12 \pm 8i$ by solving $\det(sI - (A - KC)) = (s - p_1)(s - p_2)$:

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s-1+k_1 & -2 \\ -2+k_2 & s-1 \end{bmatrix}\right)$$
$$= (s-1+k_1)(s-1) - (-2(-2+k_2))$$
$$= s^2 + s(k_1 - 2) - 3 - k_1 + 2k_2$$

Expanding the desired characteristic polynomial

$$(s-p_1)(s-p_2) = (s+12+8i)(s+12-8i) = s^2 + 24s + 64 + 144$$

and matching coefficients gives

$$\begin{cases} k_1 - 2 = 24 \\ -3 - k_1 + 2k_2 = 208 \end{cases} \Leftrightarrow \begin{cases} k_1 = 26 \\ k_2 = \dfrac{237}{2} = 118.5. \end{cases}$$

6. This problem is about compensation links.

a. The system

$$G_1(s) = \frac{1}{s(s+1)(s+0.5)}$$

is controlled with simple proportional feedback (gain 1) but does not behave as intended. Design a compensation link that makes the system 2 times faster (i.e. the crossover frequency ω_c should be doubled) while having a phase margin of $\phi_m = 12.5°$. (2.5 p)

b. Consider a phase lead compensation link and let $N \to \infty$. What kind of controller do we have now? Which problem might arise from this control structure? How can we solve this problem? (1 p)

Solution

a. From the specifications it is clear that we want to design a phase lead compensation link, having the form

$$G_r(s) = K_k N \frac{s+b}{s+bN} = K_k \frac{1+s/b}{1+s/(bN)} \qquad N > 1.$$

First we need to find our nominal crossover-frequency:

$$1 = |G_1(i\omega_c^{\text{old}})| = \frac{1}{|\omega_c^{\text{old}}|(\sqrt{(\omega_c^{\text{old}})^2+1})(\sqrt{(\omega_c^{\text{old}})^2+0.5^2})}.$$

Solving numerically, we find that $\omega_c^{\text{old}} \approx 0.815$. Thus $\omega_c^{\text{new}} \approx 1.63$.

Next, we will calculate how much the phase must increase in order to reach a phase margin of 12.5°. This given our new crossover-frequency ω_w^{new}.

$$\begin{aligned}
\Delta\phi_m &= \arg(G_r(i\omega_c^{\text{new}})) = \phi_m^{\text{new}} - (180 + \arg(G_0(i\omega_c^{\text{new}})) \\
&= 12.5 - (180 + (-90 - \frac{180}{\pi}(\arctan(\omega_c^{new}) + \arctan(\frac{\omega_c^{\text{new}}}{0.5}))) \\
&= 53.81°,
\end{aligned}$$

where ϕ_m^{new} is our desired phase margin and $\Delta\phi_m = \arg(G_r(i\omega_c^{\text{new}}))$ is the desired phase shift for the the compensation link at ω_c^{new}.

From the lecture compendium, we conclude that $N = 9$ is suitable. The next step is to find b such that the top of the phase curve for the compensation link is at ω_c^{new}. This is found from the following relationship

$$\begin{aligned}
& b\sqrt{N} = \omega_c^{new} \\
\Leftrightarrow \quad & b = \frac{\omega_c^{new}}{\sqrt{N}} = \frac{1.63}{3} = 0.542.
\end{aligned}$$

The final step is to find the gain K_k. We know from the definition of the crossover frequency that

$$1 = |G_r(i\omega_c^{\text{new}})G_1(i\omega_c^{\text{new}})| = K_k\sqrt{N}|G_1(i\omega_c^{\text{new}})|.$$

Since

$$|G_1(i\omega_c^{\text{new}})| = \frac{1}{|\omega_c^{\text{new}}|(\sqrt{(\omega_c^{\text{new}})^2+1})(\sqrt{(\omega_c^{\text{new}})^2+0.5^2})} \approx 0.1894,$$

we conclude

$$K_k = \frac{1}{0.1894\sqrt{N}} \approx 1.76.$$

b. If N goes to infinity we have designed a PD-controller, since then

$$G_r(s) = K_k \frac{1+s/b}{1+s/(bN)} \approx K_k(1+s/b).$$

A pure PD-controller can amplify noise too much due to large high frequency gain. This can be alleviated by adding a low pass filter with a relatively fast pole (faster than any zero/pole added from the PD-controller or compensation link) to the controller.