LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Automatic Control, Basic Course

Exam 23 April 2019, 8:00-13:00

Points and grades

All solutions must be well motivated. The whole exam gives 25 points. The number of points are presented after each problem. Preliminary grades:

Grade 3: at least 12 points,
4: at least 17 points,
5: at least 22 points.

Aids

Mathematical collections of formulae (e.g. TEFYMA), collections of formulae in automatic control, and calculators that are not programmed in advance.

Results

The results are presented through LADOK. Time and place for exam presentation will be announced on the course web page.

Good Luck!

Solutions to the exam in Automatic Control, Basic Course, 2019-04-23

1. A system is described by the following transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{(s+1)(s+5)}$$

a. Write the system as a differential equation. (1 p)

b. Write the system in state-space form. State the matrices A, B, and C. (1 p)

c. Is the system controllable? (1 p)

Solution

a.

$$\begin{aligned} Y(s)(s+1)(s+5) &= 2U(s) \\ s^2Y(s) + 6sY(s) + 5Y(s) &= 2U(s)\{\text{inverse Laplace transform}\} \Rightarrow \\ \ddot{y}(t) + 6\dot{y}(t) + 5y(t) &= 2u(t) \end{aligned}$$

b. We choose $x_1 = y$ och $x_2 = \dot{x}_1 = \dot{y}$ Therefore,

$$\dot{x}_2 + 6x_2 + 5x_1 = 2u$$

$$\dot{x} = \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -6x_2 - 5x_1 + 2u \end{cases} \qquad \Rightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B} u$$

$$y = x_1 \qquad \Rightarrow y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

c. The system is controllable if the controllability matrix W_s has linearly independent columns (is full rank). W_s is given by

$$W_s = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -12 \end{bmatrix}.$$

The system has linearly independent columns if the determinant W_s is nonzero:

$$\det \begin{bmatrix} 0 & 2 \\ 2 & -12 \end{bmatrix} = 0 - 4 \neq 0.$$

Therefore the system is controllable.

2. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + u^2 \\ \dot{x}_2 &= x_1^2 - x_2 + 1 \\ y &= x_1^2 + x_2 \end{aligned}$$

a. Find all the stationary points (x_1^0, x_2^0, u^0) of the system. (1 p)

b. Linearize the system for $u^0 = 1$. (2 p)

c. Is the linearized system asymptotically stable? (1 p)

Solution

To simplify the expression we introduce $f_1(x_1, x_2, u)$, $f_2(x_1, x_2, u)$ and $g(x_1, x_2, u)$ so that the system can be written as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, u) \\ y &= g(x_1, x_2, u) \end{aligned}$$

a. We find the stationary point by setting $\dot{x}_1 = \dot{x}_2 = 0$. The equation $f_1(x_1, x_2, u)) = 0$ then gives $x_1^0 = (u^0)^2$. We insert $f_2(x_1, x_2, u)) = 0$ and get $0 = (x_1^0)^2 - x_2^0 + 1$ which gives $x_2^0 = (x_1^0)^2 + 1 = (u^0)^4 + 1$. In summary, the stationery points are given by

$$(x_1^0, x_2^0, u^0) = \Big((u^0)^2, (u^0)^4 + 1, u^0\Big)$$

b. For $u^0 = 1$ the stationary point is $(x_1^0, x_2^0, u^0) = (1, 2, 1)$.

We introduce new variables $\Delta x_1 = x_1 - x_1^0$, $\Delta x_2 = x_2 - x_2^0$, $\Delta u = u - u^0$, and $\Delta y = y - y^0$. The linearized system is then given by

$$\begin{aligned} \begin{pmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{pmatrix} &= A \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + B\Delta u \\ \Delta y &= C \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + D\Delta u \end{aligned}$$

where

$$\begin{aligned} A &= \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x_1^0, x_2^0, u^0} = \begin{pmatrix} -1 & 0 \\ 2x_1^0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \\ B &= \left.\begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}\right|_{x_1^0, x_2^0, u^0} = \begin{pmatrix} 2u^0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ C &= \left.\begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}\right|_{x_1^0, x_2^0, u^0} = \begin{pmatrix} 2x_1^0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \\ D &= \left.\frac{\partial g}{\partial u}\right|_{x_1^0, x_2^0, u^0} = 0 \end{aligned}$$

c. Yes, since the A matrix has eigenvalues $\{-1, -1\}$, i.e. both lie strictly in the left half-plane.

3. Consider the Bode diagram for a transfer function is shown in Figure 1.

a. What are the poles and zeros of the transfer function? (1 p)

b. What are the phase and gain margin? (1 p)

c. Assume that we use simple feedback with a P-controller with gain $K = 1$. Is the closed-loop system stable? Motivate your asnswer. (1 p)

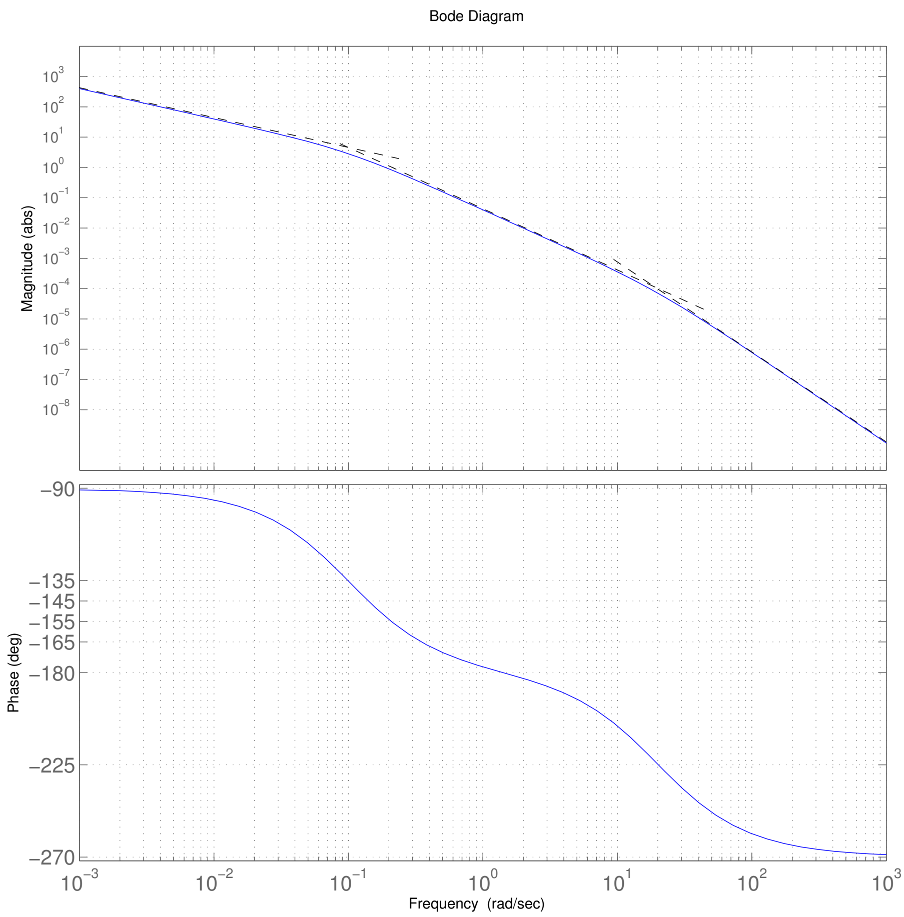

Figure 1: Bode diagram for the transfer function in problem 3.

Solution

a. In the Bode diagram we can see that the system has corner freqencies at $\omega = 20 rad/s$ och $\omega = 0.1 rad/s$. We also see that the Bode diagram has a slope of -1 and phase $= 90°$ as $\omega \to 0$. This corresponds to poles at $s = 0$, $s = -0.1$, and $s = -20$, and no zeros.

b. Phase margin $\phi_m = 27.5°$, gain margin: $g_m = 50.3$.

c. Yes, we have positive phase and amplitude margins.

4. The Nyquist diagram for a stable process is shown in figure 2. Determine with the help of the digram if each of the following statements are true, false, or if you do not have enough information about the system to answer. The system is assumed to be minimal, i.e. no pole-zero cancellation has occured. All answers must be motivated. Each correct answer gives 0.5p.

a. If the system is connected through negative feedback with a P-controller med gain $K = 2$ the system will be unstable.

b. The system's phase margin is less than $60°$.

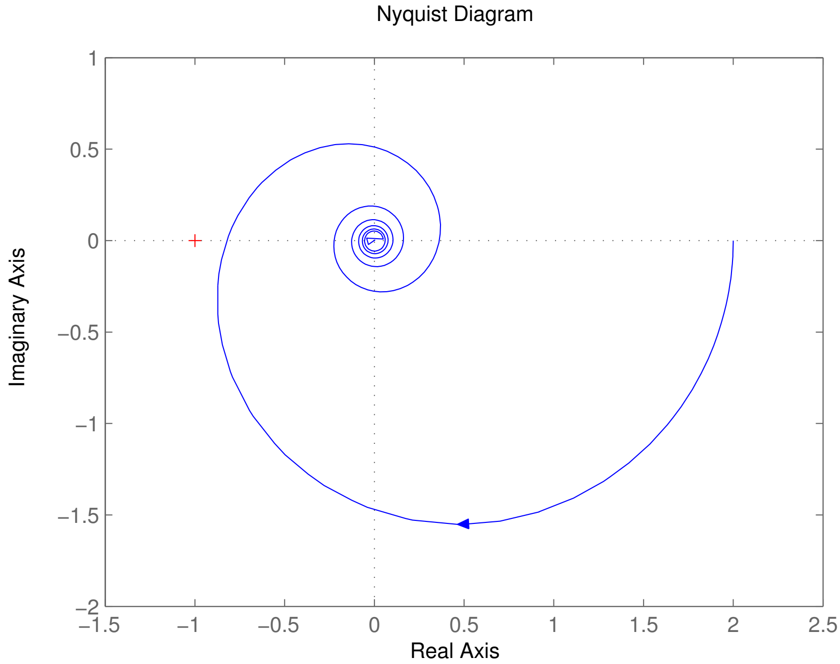

Figure 2: Nyquist diagram for the system in 4.

c. The system's lag margin is greater than 0.1 s.

d. The process' static gain is 2.

e. The process has an integrator.

f. The process is a second order system with a time delay.

Solution

a. True. The amplitude margin kan be read from the Nyquist diagram as $A_m \approx 1.2$. A P-controller with $K = 2 > A_m$ should therefore make the closed-loop system unstable.

b. True. the phase margin can be read from the diagram as $\phi_m \approx 30° < 60°$.

c. Not enough information. To calculate the systems lag margin, we must know the systems cross-over freqency, which is not visable in the diagram.

d. True. The systems stationary gain can be read from the start of the curve in the diagram, i.e. 2 where $\omega = 0$.

e. False. If the system had an integrator, the phase at low frequencies would have been $-90°$.

f. Not enough information. The time-delay makes it impossible to determine the order of the system from the Nyquist diagram.

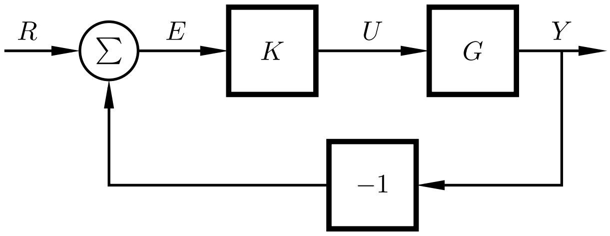

Figure 3: The closed-loop system for problems 5 and 6

5. A third order system, $G(s) = \frac{1}{s^3+2s^2+3s+1}$, is connected via negative feedback to a P-controller with gain K (see figure 3).

a. For which values of K is the closed-loop system asymptotically stable? Consider both positive and negative values of K. (2 p)

b. For which values of K is the stationary error less than 0.1 if the reference signal is a unit step? What is the lowest possible stationary error? (2 p)

Solution

a. The closed-loop system's transfer function from reference to measurement signal is $\frac{KG}{1+KG} = \frac{K}{s^3+2s^2+3s+1+K}$. The closed-loop system is asymptotically stable if all poles lie in the left half-plane (i.e. they have negative real parts). This is true only if both $a_1, a_2, a_3 > 0$ and $a_1 \cdot a_2 > a_3$ for the characteristic polynomial $s^3 + a_1 s^2 + a_2 s + a_3$. From this follows the conditions

$$
\begin{aligned}
a_1 &= 2 > 0 \text{ (OK)}, \\
a_2 &= 3 > 0 \text{ (OK)}, \\
a_3 &= 1 + K > 0 \Rightarrow K > -1 \\
a_1 \cdot a_2 &= 6 > a_3 = 1 + K \Rightarrow K < 5. \\
-1 &< K < 5
\end{aligned}
$$

b. The transfer function from reference to control error is $\frac{1}{1+KG} = \frac{s^3+2s^2+3s+1}{s^3+2s^2+3s+1+K}$ and has the same poles as the transferfunction from reference to measurement signal. We can use the final value theorum is the system is stable ($-1 < K < 5$).

$$
\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^3+2s^2+3s+1}{s^3+2s^2+3s+1+K} = \frac{1}{1+K} < 0.1 \Rightarrow K > 9
$$

Therefore, there is no K that gives a stationary error of less than 0.1.
The lowest possible stationary error is $\lim_{K\to 5} \frac{1}{1+K} = \frac{1}{6}$.

6. A system has the transfer function

$$
G(s) = \frac{1}{s(s+1)(s+2)},
$$

and is controlled by a P-controller, K (see figure 3).

a. Set $K = 1$ (which gives a stable system) and assume that

$$r(t) = \begin{cases} t, \ t \geq 0 \\ 0, \ t < 0 \end{cases}$$

Design a compensation link

$$G_k(s) = M\frac{s+a}{Ms+a} \tag{1}$$

so that the stationary error is reduced by a factor of 3, without reducing the phase margine by more than 6°. Show the equation for the cross-over freqency (ω_c) of $G(s)$ and show that $\omega_c \approx 0.45$. (2 p)

b. Assume that you want the compensation link to have even less impact on the phase margin, compared to your design in a). How would you need to *change* M and/or a in (1)?
Note: you do not need to calculate new parameters, only clarify if you would increase or decrease them and why. (1 p)

Solution

a. To minimize the stationery error by a factor of 3 we must choose $M = 3$. The uncompensated transfer function is

$$G(s) = \frac{1}{s(s+1)(s+2)}.$$

The cross-over frequency ω_c is given by

$$\begin{aligned} |G_o(i\omega_c)| &= 1 \\ |G_o(i\omega_c)| &= \frac{1}{\omega_c\sqrt{1+\omega_c^2}\sqrt{4+\omega_c^2}} \\ \omega_c^2(1+\omega_c^2)(4+\omega_c^2) &= 1 \\ \omega_c &\approx 0.45. \end{aligned}$$

According to the rule-of-thumb the phase margin does not decrease with more than 6° if we choose $a = 0.1\omega_c = 0.045$. We thus get the transfer function for the compensation link as

$$G_k(s) = M\frac{s+a}{Ms+a} = 3\frac{s+0.045}{3s+0.045}$$

b. M could have been decreased to diminish the effect on the phase margin, but M must not be changed due to the constraint to decrease the stationary error. The only other possibility is to decrease a. The reason is that for each $\omega > 0$ we have that:

$$\lim_{a\to 0} |G_k(i\omega)| = 1 \quad \text{och} \quad \lim_{a\to 0} \arg\{G_k(i\omega)\} = 0.$$

and thus we can decrease the influence of G_k on the phase margin arbitrarily low by just decreasing a enough (see figure 11.6 in Lecture Notes in Automatic Control, Tore Hägglund, 2017)

7. Consider
$$\dot{x} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u.$$

a. Determine the poles for the (open-loop) system. (1 p)

b. Determine a control law
$$u = -Lx,$$
such that the closed-loop system poles will be twice as fast as the open-loop system's fastest pole. (2 p)

c. Assume that we cannot measure all states, but that we access to an output signal
$$y = Cx = [c_1 \; c_2]x.$$
We want to use the output signal y to estimate the two states of the system $x = [x_1, x_2]$. What constraints does this impose on c_1 and c_2? (2 p)

Solution

a. The characteristic polynomial
$$\det(sI - A) = (s+1)(s+3),$$
gives the two poles $s = -1$ and $s = -3$.

b. The fastest pole is the one situated furthest from the origin, i.e., $s = -3$. We thus need to find L such that the two poles are placed at the distance 6 from the origin, given e.g., by double poles in $s = -6$. The characteristic polynomial for the closed-loop system will then be
$$(s+6)^2 = s^2 + 12s + 36.$$
We have that
$$\det(sI - A + BL) = s^2 + (4 + 4l_2)s + 8l_1 + 4l_2 + 3,$$
and thus
$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 25/8 \\ 2 \end{bmatrix}^{\mathrm{T}}.$$

c. The observability matrix is given by
$$W_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ -c_1 & 2c_1 - 3c_2 \end{bmatrix}.$$
The system is observable if W_0 has full rank, i.e.,
$$0 \neq \det \begin{bmatrix} c_1 & c_2 \\ -c_1 & 2c_1 - 3c_2 \end{bmatrix} = c_1(2c_1 - 3c_2) + c_1 c_2 = 2c_1(c_1 - c_2).$$
Thus, we must have that
$$\begin{cases} c_1 \neq 0 \\ c_1 \neq c_2 \end{cases}.$$