LUND UNIVERSITY

Robust Control

Lecture 4

Dongjun Wu

H_∞-synthesis problem

Recall the DK-iteration

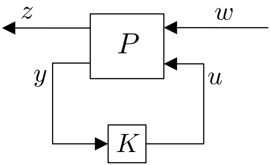
- K**-step**: fix $D(s)$, solve (H_∞-synthesis)

$$\min_K \left\| F_\ell\left(\begin{bmatrix} D & \\ & I \end{bmatrix} P \begin{bmatrix} D & \\ & I \end{bmatrix}^{-1}, K \right) \right\|_\infty$$

- D**-step**: fix $K(s)$, solve frequency-wise (convex program)

$$\min_{D \in \mathcal{D}_\Pi, D, D^{-1} \in H_\infty} \bar{\sigma}(D F_\ell(P,K) D^{-1}(j\omega))$$

H_∞-synthesis problem



Optimal H_∞ control:

$$\min_K \|F_\ell(P,K)\|_\infty.$$

Suboptimal H_∞ control: given $\gamma > 0$, find K such that

$$\|F_\ell(P,K)\|_\infty < \gamma.$$

- Riccati equation approach
- Optimization approach
- Model matching approach

The "DGKF" solution

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 34, NO. 8, AUGUST 1989 831

State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

JOHN C. DOYLE, KEITH GLOVER, MEMBER, IEEE, PRAMOD P. KHARGONEKAR, MEMBER, IEEE, AND BRUCE A. FRANCIS, FELLOW, IEEE

***Abstract*—Simple state-space formulas are derived for all controllers solving a standard $\mathcal{H}_\infty$ problem: for a given number $\gamma > 0$, find all controllers such that the $\mathcal{H}_\infty$ norm of the closed-loop transfer function is (strictly) less than γ. A controller exists if and only if the unique stabilizing solutions to two algebraic Riccati equations are positive definite and the spectral radius of their product is less than γ^2. Under these conditions, a parametrization of all controllers solving the problem is given as a linear fractional transformation (LFT) on a contractive, stable free parameter. The state dimension of the coefficient matrix for the LFT, constructed using these same two Riccati solutions, equals that of the plant, and has a separation structure reminiscent of classical LQG (i.e., $\mathcal{H}_2$) theory. This paper is also intended to be of tutorial value, so a standard $\mathcal{H}_2$ solution is developed in parallel.**

including disturbances, sensor noise, and commands; the output z is an error signal; y is the measured variables; and u is the control input. The diagram is also referred to as a linear fractional transformation (LFT) on K, and G is called the coefficient matrix for the LFT. The resulting closed-loop transfer function from w to z is denoted by T_{zw}.

The main $\mathcal{H}_\infty$ output feedback results of this paper as described in the Abstract are presented in Section III. The proofs of these results in Section V exploit the "separation" structure of the controller, which is reminiscent of the classical $\mathcal{H}_2$ controller. Of course, there are significant differences that reflect the fact that the $\mathcal{H}_\infty$ criterion corresponds to designing for the worst exogenous signal. These are also discussed in Section V. Special attention will be given to the central controller, obtained by setting

The "DGKF" solution

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \qquad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$

Assumptions:

- A1 (A, B_1) stabilizable, (C_1, A) detectable;
- A2 (A, B_2) stabilizable, (C_2, A) detectable;
- A3 $D_{12}^\top [C_1 \quad D_{12}] = [0 \quad I]$;
- A4 $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^\top = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Theorem

There exists a controller such that $\|T_{zw}\|_\infty < \gamma$ iff:

1. $X_\infty \geq 0$ *is a solution to the ARE*
$$A^\top X_\infty + X_\infty A + X_\infty(\gamma^{-2} B_1 B_1^\top - B_2 B_2^\top) X_\infty + C_1^\top C_1 = 0.$$
2. $Y_\infty \geq 0$ *is a solution to the ARE*
$$A Y_\infty + Y_\infty A^\top + Y_\infty(\gamma^{-2} C_1^\top C_1 - C_2^\top C_2) Y_\infty + B_1 B_1^\top = 0.$$
3. $\rho(X_\infty Y_\infty) < \gamma^2$.

The "DGKF" solution

Theorem (continued)

Moreover, when these conditions hold, all such controllers are given by $K = F_\ell(K_c, Q)$ *where*

$$K_c = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

$$F_\infty = -B_2^\top X_\infty$$

$$L_\infty = -Y_\infty C_2^\top$$

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

$$A_\infty = A + \gamma^{-2} B_1 B_1^\top X_\infty + B_2 F_\infty$$

and $Q(s)$ *is any stable proper transfer matrix s.t.* $\|Q\|_\infty < \gamma$*. For* $Q = 0$*, we get*

$$K(s) = -F_\infty (sI - A_\infty)^{-1} Z_\infty L_\infty.$$

Algorithm

Given γ, test if the conditions of the theorem are satisfied; if yes, decrease γ; otherwise increase. Proceed using bisection.

Differential game and worst case optimal control

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x \qquad + D_{12} u$$

Consider the problem

$$\min_{u \in L_2} \max_{w \in L_2} \frac{1}{2} \int_0^\infty \|z\|_2^2 - \gamma^2 \|w\|_2^2 \mathrm{d}t$$

$$\int_0^\infty \|z\|_2^2 - \gamma^2 \|w\|^2 \mathrm{d}t = \int_0^\infty \|C_1 x + D_{12} u\|^2 - \gamma^2 \|w\|^2 \mathrm{d}t$$
$$\overset{\text{by A3}}{=} \int_0^\infty x^\top C_1^\top C_1 x + \|u\|^2 - \gamma^2 \|w\|^2 \mathrm{d}t$$

$$\min_{u \in L_2} \max_{w \in L_2} \frac{1}{2} \int_0^\infty x^\top C_1^\top C_1 x + \|u\|^2 - \gamma^2 \|w\|^2 \mathrm{d}t$$

subject to

$$\dot{x} = Ax + B_1 w + B_2 u$$

Worst-case optimal control

$$\min_{u\in L_2}\max_{w\in L_2}\frac{1}{2}\int_0^\infty x^\top C_1^\top C_1 x + \|u\|^2 - \gamma^2\|w\|^2 \mathrm{d}t$$

subject to

$$\dot{x} = Ax + B_1 w + B_2 u$$

Define the Hamiltonian

$$H(x,p,u,w) = p^\top(Ax + B_1 w + B_2 u) + \frac{1}{2}\left(x^\top C_1^\top C_1 x + \|u\|^2 - \gamma^2\|w\|^2\right)$$

Let $V(x) = \frac{1}{2}x^\top X_\infty x$ (x is the IC), with $X_\infty \geq 0$ be the value function, then by Bellman's principle of optimality

$$0 = \min_u \max_w H(x, V_x, u, w)$$

$$\implies A^\top X_\infty + X_\infty A + C_1^\top C_1 + X_\infty(\gamma^{-2}B_1 B_1^\top - B_2 B_2^\top)X_\infty = 0$$

and the optimal control

optimal control:	$u = -B_2^\top X_\infty x := F_\infty x$
worst case disturbance:	$w = \gamma^{-2}B_1^\top X_\infty x$

Pure state-feedback control. However, we need output feedback, i.e., $u = F(y(\cdot))$.

H_∞-optimal observer

$$\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x \qquad\quad + D_{12} u \\
y &= C_2 x + D_{21} w
\end{aligned}$$

Luenberger observer:

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B_2 u + L(\hat{y} - y) \\
\hat{y} &= C_2 \hat{x} \\
\hat{z} &= C_1 \hat{x} \qquad\quad + D_{12} u
\end{aligned}$$

(w is not measurable!)

Error system, $\xi = x - \hat{x}$, error output $e = z - \hat{z}$.

$$\begin{aligned}
\dot{\xi} &= (A + LC_2)\xi + (B_1 + LD_{21}) w \\
e &= C_1 \xi
\end{aligned}$$

$$\min_L \max_w \frac{1}{2} \int_0^\infty \|e\|^2 - \gamma^2 \|w\|^2 \mathrm{d}t$$

H_∞-optimal observer

H_∞-opt. obs.: $\begin{aligned} \dot{\xi} &= (A+LC_2)\xi + (B_1+LD_{21})w \\ e &= C_1\xi \end{aligned}$, $\min_L \max_w \frac{1}{2}\int_0^\infty \|e\|^2 - \gamma^2\|w\|^2 \mathrm{d}t$

H_∞-opt. contr.: $\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \qquad + D_{12} u \end{aligned}$, $\min_{u\in L_2} \max_{w\in L_2} \frac{1}{2}\int_0^\infty \|z\|_2^2 - \gamma^2\|w\|_2^2 \mathrm{d}t$

$\Updownarrow$ (optimal u is shown to be state-feedback)

$\begin{aligned} \dot{x} &= (A+B_2K)x + B_1 w \\ z &= (C_1+D_{12}K)x \end{aligned}$, $\min_{u\in L_2} \max_{w\in L_2} \frac{1}{2}\int_0^\infty \|z\|_2^2 - \gamma^2\|w\|_2^2 \mathrm{d}t$

Note

$$T_{zw} = (C_1 + D_{12}K)(sI - A - B_2K)^{-1}B_1$$

and

$$T_{ew} = C_1(sI - A - LC_2)^{-1}(B_1 + LD_{21})$$

$$\Rightarrow T_{ew}^\top = (B_1^\top + D_{21}^\top L^\top)(sI - A^\top - C_2^\top L^\top)C_1^\top$$

Comparing $T_{ew}^\top$ to T_{zw}, we get:

H_∞-optimal observer

H_∞-optimal observer gain

$$L_\infty := -Y_\infty C_2^\top$$

where $Y_\infty \geq 0$ solves

$$AY_\infty + Y_\infty A^\top + B_1 B_1^\top + Y_\infty(\gamma^{-2} C_1^\top C_1 - C_2^\top C_2) Y_\infty = 0.$$

H_∞ observer-based controller

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \qquad\qquad + D_{12} u \\ y &= C_2 x + D_{21} w\end{aligned}$$

H_∞ observer

$$\dot{\hat{x}} = A\hat{x} + B_1 \hat{w} + B_2 u + Z_\infty L_\infty (\hat{y} - y)$$

where

$$\underbrace{u = F_\infty \hat{x}}_{\text{worst case optimal control}} \qquad \underbrace{\hat{w} = \gamma^{-2} B_1^\top X_\infty \hat{x}}_{\text{worst cased disturbance}} \qquad \underbrace{Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}}_{\text{coupling compensation}}$$

The "DGKF" solution

Theorem (continued)

There exists a controller such that $\|T_{zw}\|_\infty < \gamma$ *iff:*

1. $X_\infty \geq 0$ *is a solution to the ARE*
$$A^\top X_\infty + X_\infty A + X_\infty(\gamma^{-2}B_1B_1^\top - B_2B_2^\top)X_\infty + C_1^\top C_1 = 0.$$
2. $Y_\infty \geq 0$ *is a solution to the ARE*
$$AY_\infty + Y_\infty A^\top + Y_\infty(\gamma^{-2}C_1^\top C_1 - C_2^\top C_2)Y_\infty + B_1B_1^\top = 0.$$
3. $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, when these conditions hold, all such controllers are given by $K = F_\ell(K_c, Q)$ *where*

$$K_c = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

$$F_\infty = -B_2^\top X_\infty, \quad L_\infty = -Y_\infty C_2^\top, \quad Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1},$$
$$A_\infty = A + \gamma^{-2}B_1B_1^\top X_\infty + B_2F_\infty$$

and $Q(s)$ *is any stable proper transfer matrix s.t.* $\|Q\|_\infty < \gamma$. *For* $Q = 0$, *we get* $K(s) = -F_\infty(sI - A_\infty)^{-1}Z_\infty L_\infty$.

Youla Parametrization

- $M, N \in RH_\infty$ are *right coprime* (over RH_∞) if $\exists X_r, Y_r \in RH_\infty$, s.t. $X_r M + Y_r N = I$.
- $\tilde{M}, \tilde{N} \in RH_\infty$ are *left coprime* (over RH_∞) if $\exists X_l, Y_l \in RH_\infty$, s.t. $\tilde{M} X_l + \tilde{N} Y_l = I$.
- Let P be real rational.
  - *right coprime factorization of P:* $P = NM^{-1}$, M, N right coprime;
  - *left coprime factorization of P:* $P = \tilde{M}^{-1}\tilde{N}$. $\tilde{M}, \tilde{N}$ left coprime;
  - *double coprime factorization of P:* if $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$, and X_r, Y_r, X_l, Y_l s.t.

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I.$$

- Let $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be proper, rational, then

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right], \quad \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right]$$

  is a double coprime factorization of P, where F, L are s.t. $A+BF$ and $A+LC$ are stable.

Youla Parameterization: all stabilizing controllers

Theorem

Consider $F_\ell(P,K)$. Let

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I$$

be a double coprime factorization of P_{22}. Then the set of all proper controllers achieving internal stability is parameterized by

$$K = F_\ell(K_c, Q)$$

where

$$K_c = \begin{bmatrix} -Y_l X_l^{-1} & X_r^{-1} \\ X_l^{-1} & -X_l^{-1} N \end{bmatrix} = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right]$$

and $Q \in RH_\infty$ is s.t. $(I + V_0^{-1}NQ)(j\infty)$ is invertible ($A + B_2F$ and $A + LC_2$ stable).

Note: closed loop system:

$$F_\ell(P, F_\ell(K_c, Q)) = F_\ell(P \star K_c, Q).$$

Redheffer start products: $P \star K$, see [Chapter 9, ZD].

The Kalman–Yakubovich–Popov (KYP) lemma

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

The following are equivalent:

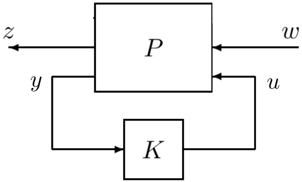
1. $\|G\|_\infty < \gamma$ and A Hurwitz.
2. There exists $X > 0$ such that

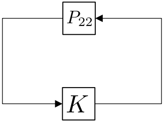
$$\begin{bmatrix} A^\top X + XA & XB \\ B^\top X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0.$$

3. There exists $X > 0$ such that

$$\begin{bmatrix} A^\top X + XA & XB & C^\top \\ B^\top X & -\gamma I & D^\top \\ C & D & -\gamma I \end{bmatrix} < 0.$$

Recall of state-space LFT





$$\dot{x} = Ax + B_1 w + B_2 u \qquad \dot{x}_K = A_K x_K + B_K y$$
$$z = C_1 x + D_{11} w + D_{12} u \qquad u = C_K x_K + D_K y$$
$$y = C_2 x + D_{21} w + D_{22} u$$

$$\begin{aligned} \dot{x}_{\mathrm{cl}} &= A_{\mathrm{cl}} x_{\mathrm{cl}} + B_{\mathrm{cl}} w \\ z &= C_{\mathrm{cl}} x_{\mathrm{cl}} + D_{\mathrm{cl}} w \end{aligned}, \quad x_{\mathrm{cl}} = \begin{bmatrix} x \\ x_K \end{bmatrix}$$

Assume $D_{22} = 0$, then

$$\begin{bmatrix} A_{\mathrm{cl}} & B_{\mathrm{cl}} \\ C_{\mathrm{cl}} & D_{\mathrm{cl}} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

The closed-loop parameters are *affine* in controller parameters.

Application of KYP

By KYP lemma

$$\|F_\ell(P,K)\|_\infty < \gamma, \quad A_{\rm cl} \text{ Hurwitz}$$

$$\Updownarrow$$

$$\exists X_{\rm cl} > 0 \text{ such that } \begin{bmatrix} A_{\rm cl}^\top X_{\rm cl} + X_{\rm cl} A_{\rm cl} & X_{\rm cl} B_{\rm cl} & C_{\rm cl}^\top \\ B_{\rm cl}^\top X_{\rm cl} & -\gamma I & D_{\rm cl}^\top \\ C_{\rm cl} & D_{\rm cl} & -\gamma I \end{bmatrix} < 0.$$

Goal: find A_K, B_K, C_K, D_K such that the above conditions hold.

Note:

- A_K, B_K, C_K, D_K fixed: SDP
- $X_{\rm cl}$ fixed: SDP
- Not jointly convex

A simpler case: state feedback

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u\end{aligned}$$

state feedback $u = D_K x$:

$$\begin{aligned}\dot{x} &= (A + B_2 D_K)x + B_1 w \\ z &= (C_1 + D_{12} D_K)x + D_{11} w\end{aligned}$$

$$\begin{bmatrix} A_{\mathrm{cl}}^\top X_{\mathrm{cl}} + X_{\mathrm{cl}} A_{\mathrm{cl}} & X_{\mathrm{cl}} B_{\mathrm{cl}} & C_{\mathrm{cl}}^\top \\ B_{\mathrm{cl}}^\top X_{\mathrm{cl}} & -\gamma I & D_{\mathrm{cl}}^\top \\ C_{\mathrm{cl}} & D_{\mathrm{cl}} & -\gamma I \end{bmatrix} < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} (A + B_2 D_K)^\top X_{\mathrm{cl}} + X_{\mathrm{cl}}(A + B_2 D_K) & X_{\mathrm{cl}} B_1 & (C_1 + D_{12} D_K)^\top \\ B_1^\top X_{\mathrm{cl}} & -\gamma I & D_{11}^\top \\ C_1 + D_{12} D_K & D_{11} & -\gamma I \end{bmatrix} < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} X_{cl}^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} (A + B_2 D_K)^\top X_{\mathrm{cl}} + X_{\mathrm{cl}}(A + B_2 D_K) & X_{\mathrm{cl}} B_1 & (C_1 + D_{12} D_K)^\top \\ B_1^\top X_{\mathrm{cl}} & -\gamma I & D_{11}^\top \\ C_1 + D_{12} D_K & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} X_{cl}^{-1} & & \\ & I & \\ & & I \end{bmatrix} < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} X_{cl}^{-1}(A + B_2 D_K)^\top + (A + B_2 D_K) X_{cl}^{-1} & B_1 & X_{cl}^{-1}(C_1 + D_{12} D_K)^\top \\ B_1^\top & -\gamma I & D_{11}^\top \\ (C_1 + D_{12} D_K) X_{cl}^{-1} & D_{11} & -\gamma I \end{bmatrix} < 0$$

A simpler case: state feedback

$$\begin{bmatrix} X_{cl}^{-1}(A+B_2D_K)^\top + (A+B_2D_K)X_{cl}^{-1} & B_1 & X_{cl}^{-1}(C_1+D_{12}D_K)^\top \\ B_1^\top & -\gamma I & D_{11}^\top \\ (C_1+D_{12}D_K)X_{cl}^{-1} & D_{11} & -\gamma I \end{bmatrix} < 0$$

$\Updownarrow$ (letting $\bar{D}_K =: D_K X_{cl}^{-1}$)

$$\begin{bmatrix} (AX_{cl}^{-1}+B_2\bar{D}_K)^\top + AX_{cl}^{-1} + B_2\bar{D}_K & B_1 & (C_1X_{cl}^{-1}+D_{12}\bar{D}_K)^\top \\ B_1^\top & -\gamma I & D_{11}^\top \\ C_1X_{cl}^{-1}+D_{12}\bar{D}_K & D_{11} & -\gamma I \end{bmatrix} < 0$$

affine in the variable $(X_{cl}^{-1}, \bar{D}_K)$, i.e.g, LMI! Once $(X_{cl}^{-1}, \bar{D}_K)$ has been found,

$$D_K = \bar{D}_K X_{cl}.$$

Exercise

$$\dot{x} = Ax + B_1 w + B_2 u \qquad \dot{x}_K = A_K x_K + B_K y$$

$$z = C_1 x + D_{11} w + D_{12} u \qquad u = C_K x_K + D_K y$$

$$y = C_2 x + D_{21} w + D_{22} u$$

Derive the LMI formulation of

$$\min_{(A_K, B_K, C_K, D_K)} \|T_{zw}\|_\infty.$$

Model matching approach

Model matching problem: given $T_1, T_2, T_3 \in RH_\infty$

$$\min_{Q \in RH_\infty} \|T_1 + T_2 Q T_3\|_\infty$$

Let

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I$$

be a double coprime factorization of P_{22}.

$$F_\ell(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Youla parameterization: $K = (MQ - Y_l)(NQ + X_l)^{-1}$

$$\begin{aligned} \implies K(I - P_{22}K)^{-1} &= K[I - \tilde{M}^{-1}\tilde{N}(MQ - Y_l)(NQ + X_l)^{-1}]^{-1} \\ &= (MQ - Y_l)\tilde{M} \end{aligned}$$

$$\begin{aligned} \implies F_\ell(P,K) &= P_{11} + P_{12}(MQ - Y_l)\tilde{M}P_{21} \\ &= \underbrace{(P_{11} - P_{12}Y_l\tilde{M}P_{21})}_{T_1} + \underbrace{P_{12}M}_{T_2}\,Q\,\underbrace{\tilde{M}P_{21}}_{T_3}. \end{aligned}$$

Exercise: write a program to solve the problem.

H_2-optimal control

$$\min_K \|F_\ell(P,K)\|_2$$

H_2-norm:

$$\|G\|_2 := \mathrm{tr}E[z(t)z^\top(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}[G(j\omega)G(j\omega)^*]\mathrm{d}\omega$$

where w is a white noise of unit intensity.

LQR

$$\min_u \int_0^\infty \|z(t)\|^2 \mathrm{d}t \implies u = -B_2^\top X_\infty x$$

$$X_\infty \quad \leftarrow \quad A^\top X_\infty + X_\infty A + C_1^\top C_1 - X_\infty B_2 B_2^\top X_\infty = 0$$

Kalman-Bucy filter:

$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(y - C_2\hat{x}) \implies L = -Y_\infty C_2^\top$$

$$Y_\infty \quad \leftarrow \quad AY_\infty + Y_\infty A^\top + B_1 B_1^\top - Y_\infty C_2^\top C_2 Y_\infty = 0$$

Observer-based controller $u = -B_2^\top X_\infty \hat{x}$.

H_2 is the limit of H_∞ as $\gamma \to \infty \implies$ no robustness margin.