LUND UNIVERSITY

Robust Control

Lecture 3

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Review of last lecture

- How to model uncertainties:
  - Real parameters uncertainties. e.g., $|\delta_i| \le 1$.
  - Complex disk uncertainties: e.g. $\tilde{P} = (I + W_1 \Delta W_2)P$, with $\|\Delta\|_\infty \le 1$. E.g., $\tilde{P} = (1 + w\Delta)P$,

$$|w(j\omega)| \ge \left| \frac{\tilde{P}(j\omega) - P(j\omega)}{P(j\omega)} \right|.$$

- Nominal performance specifications:
  - Achieve high loop and controller gain in the necessary frequency range.
  - Weighted H_∞-performance, e.g. $\|W_e S\|_\infty \le 1$, or more generally

$$\|F_\ell(P, K)\|_\infty \le 1$$

  - Synthesis problem:

$$\min_K F_\ell(P, K), \text{ or find } K \text{ s.t. } \|F_\ell(P, K)\|_\infty \le 1$$

    for a generalized plant P.

This lecture

- Stability & performance specifications in the presence of uncertainties
- Structural uncertainties

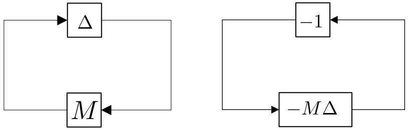
Robust stability and performance

Definition

Given the description of an uncertainty model Π, and a set of performance objectives. Suppose $P_0 \in \Pi$ is the nominal model and K the resulting controller. Then the closed-loop system is said to have

- **Nominal stability (NS):** if K internally stabilizes P_0.
- **Robust stability (RS):** if K internally stabilizes every $P \in \Pi$.
- **Nominal performance (NP):** if the performance objectives are satisfied for P_0.
- **Robust performance (RP):** if the performance objectives are satisfied for every $P \in \Pi$.

Robust stability



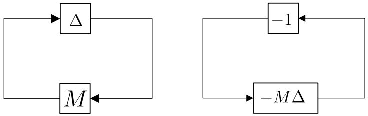
If M and Δ are stable, then under what condition is the interconnection internally stable?

Theorem (Nyquist theorem)

Assume that the realization of L is stabilizable+detectable. Let N denote the number of unstable poles of L. Then the negative feedback system is internally stable iff the Nyquist plot of $\det(I + L(s))$

- *makes N anti-clockwise encirclements of the origin, and*
- *does not pass through the origin.*

Robust stability: real & complex uncertainty



Theorem

Assume M and Δ are stable, and that the uncertainty set $\boldsymbol{\Delta}$ satisfies

$$\Delta \in \boldsymbol{\Delta} \implies \epsilon\Delta \in \boldsymbol{\Delta} \text{ for all } \epsilon \in [0,1]$$

Then the interconnection is internally stable if and only if

- *the Nyquist plot of $\det(I - M\Delta)(s)$ does not pass nor encircle the origin for any $\Delta \in \boldsymbol{\Delta}$,*
- *or equivalently*

$$\det(I - M\Delta)(j\omega) \neq 0,\ \forall \omega \in \mathbb{R} \cup \{\infty\},\ \forall \Delta \in \boldsymbol{\Delta}.$$

($\Rightarrow$) obvious.

($\Leftarrow$) note that for $\Delta = 0$, $\det(I - M\Delta) = 1$. If $\exists \Delta' \in \boldsymbol{\Delta}$ such that the Nyquist plot of $\det(I - M\Delta')(s)$ encircles the origin, then as $\epsilon \to 0$, $\det(I - \epsilon M\Delta')$ must pass through the origin for some $\omega \in \mathbb{R} \cup \{\infty\}$. But $\epsilon\Delta' \in \boldsymbol{\Delta}$.

Robust stability: complex uncertainty

Theorem

Assume M and Δ are stable, and that the uncertainty set $\boldsymbol{\Delta}$ satisfies $0 \in \boldsymbol{\Delta}$,

$$\Delta \in \boldsymbol{\Delta} \implies \epsilon\Delta \in \boldsymbol{\Delta} \text{ for any complex } |\epsilon| \leq 1.$$

The interconnection is internally stable if and only if

$$\rho(M\Delta(j\omega)) < 1, \quad \forall\omega \in \mathbb{R} \cup \{\infty\}, \forall\Delta \in \boldsymbol{\Delta}.$$

($\Leftarrow$): $\det(I - M\Delta)(j\omega)) = \prod_i (j\omega - \lambda_i(M\Delta(j\omega))) \neq 0$.

($\Rightarrow$): If $\exists\Delta'$, s.t. $\rho(M\Delta'(j\omega)) = 1$ at some frequency. Then $|\lambda_i(M\Delta')| = 1$ some some i. Now

$$\det(I - M\epsilon\Delta') = \prod_i (1 - \lambda_i(\epsilon M\Delta'))$$

thus one can choose $\epsilon = \exp(-j\angle\lambda_i(M\Delta'))$ to make $\det(I - M\epsilon\Delta') = 0$.

Disk uncertainty

Lemma

Let D be the set of all transfer matrices satisfying $\bar{\sigma}(\Delta) \leq 1$. Then

$$\max_{\Delta \in D} \rho(M\Delta) = \max_{\Delta \in D} \bar{\sigma}(M\Delta) = \max_{\Delta \in D} \bar{\sigma}(\Delta)\bar{\sigma}(M) = \bar{\sigma}(M).$$

Proof:

$$\rho(M\Delta) \leq \bar{\sigma}(M\Delta) \leq \bar{\sigma}(M)\bar{\sigma}(\Delta)$$

Reverse direction: $M = U\Sigma V^*$, choose $\Delta = VU^*$. Then

$$\rho(M\Delta) = \rho(U\Sigma V^* VU^*) = \rho(U\Sigma U^*) = \rho(\Sigma) = \bar{\sigma}(M).$$

Theorem

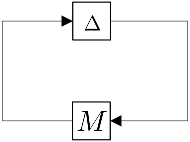
Assume M is stable. Then the interconnection is internally stable for all Δ satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$[\bar{\sigma}(M(j\omega)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}] \iff \|M\|_\infty < 1$$

The interconnection is internally stable for all Δ satisfying $\|\Delta\|_\infty < 1$ if and only if

$$[\bar{\sigma}(M(j\omega)) \leq 1, \quad \forall \omega \in \mathbb{R}] \iff \|M\|_\infty \leq 1.$$

Corollary: small gain theorem



Corollary

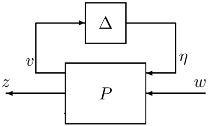
Assume M is stable. Then the interconnection is internally stable for all Δ satisfying $\|\Delta\|_\infty \leq \frac{1}{\gamma}$ if and only if

$$\|M\|_\infty < \gamma$$

The interconnection is internally stable for all Δ satisfying $\|\Delta\|_\infty < \frac{1}{\gamma}$ if and only if

$$\|M\|_\infty \leq \gamma.$$

Robust stability



Corollary

Assume that P_{11} is stable[a]*. Then the $\mathcal{U}$LFT is internally stable*

- *for all* $\|\Delta\|_\infty \leq \frac{1}{\gamma}$ *if and only if* $\|P_{11}\| < \gamma$,
- *for all* $\|\Delta\|_\infty < \frac{1}{\gamma}$ *if and only if* $\|P_{11}\| \leq \gamma$.

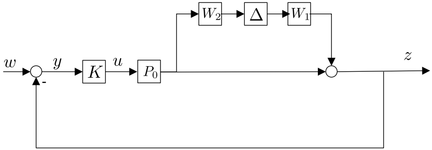
[a] We always tacitly assume that it has a stabilizable+detectable realization

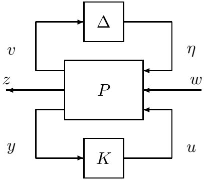
Example

Consider the multiplicative uncertainty

$$\Pi = (I + W_1 \Delta W_2) P_0, \; \|\Delta\|_\infty \leq 1$$

Suppose that K nominally stabilizes P_0.





$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 & W_2 P_0 \\ W_1 & 0 & P_0 \\ -W_1 & I & -P_0 \end{bmatrix} \begin{bmatrix} \eta \\ w \\ u \end{bmatrix}$$

$$F_\ell(P,K) = \begin{bmatrix} 0 & 0 \\ W_1 & 0 \end{bmatrix} + \begin{bmatrix} W_2 P_0 \\ P_0 \end{bmatrix} K (I + P_0 K)^{-1} \begin{bmatrix} -W_1 & I \end{bmatrix}$$

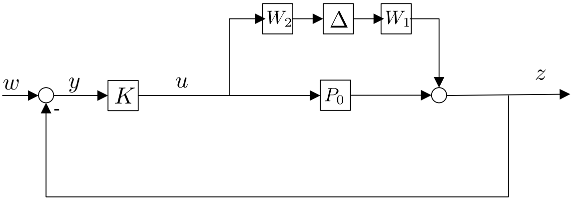
The system is internally stable if and only

$$\| W_2 T_o W_1 \|_\infty < 1.$$

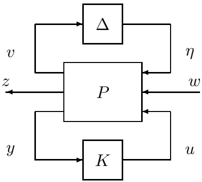
Class exercise

Find the robust stability criterion for additive uncertainty:

$$\tilde{P} = P + W_1 \Delta W_2, \quad \|\Delta\|_\infty \leq 1$$



Summary



If K internally stabilizes P_{22} by positive feedback, then the system is internally stable for all $\|\Delta\|_\infty \leq 1$ if and only if

$$\|(F_\ell(P,K))_{1,1}\|_\infty < 1.$$

Robust performance

Recall the nominal performance condition[1]

$$\|F_\ell(M,K)\|_\infty < 1 \tag{1}$$

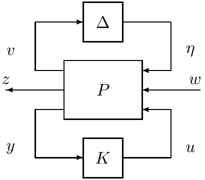
or

$$\min_K \|F_\ell(M,K)\|_\infty.$$

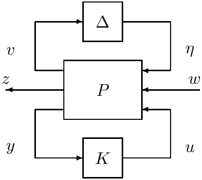
When uncertainty is introduced, robust performance requires

- robust stability: $F_u(F_\ell(P,K),\Delta)$ is internally stable,
- robust performance: $\|F_u(F_\ell(P,K),\Delta)\|_\infty < 1$

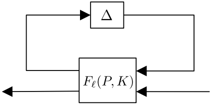
for all $\|\Delta\|_\infty \leq 1$. Note that robust stability is guaranteed by $\|F_\ell(P,K)_{1,1}\| < 1$.



Robust performance

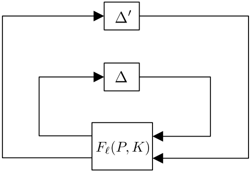


$$\|F_u(F_\ell(P,K),\Delta)\|_\infty < 1$$



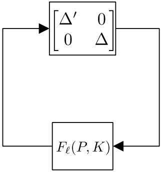
$\|F_u(F_\ell(P,K),\Delta)\|_\infty < 1$

$\forall \|\Delta\|_\infty \leq 1$



robust stability

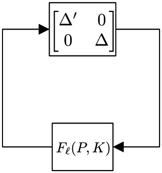
$\forall \|\Delta'\|_\infty \leq 1,\ \|\Delta\|_\infty \leq 1$



robust stability

$\forall \Delta'' = \begin{bmatrix}\Delta' & 0\\ 0 & \Delta\end{bmatrix}$

Structured uncertainty



robust stability

$$\forall \Delta'' = \begin{bmatrix} \Delta' & 0 \\ 0 & \Delta \end{bmatrix}$$

$$\|\Delta'\|_\infty \le 1, \ \|\Delta\|_\infty \le 1$$

Warning!

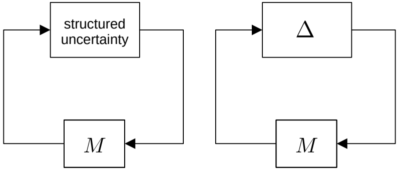
The uncertainty is not of a disk type! So it is **not** equivalent to requiring $\|F_\ell(P,K)\|_\infty < 1$!

More generally, we may consider the "structured uncertainty":

$$\Delta = \begin{bmatrix} \delta_1 I & & & & & \\ & \ddots & & & & \\ & & \delta_s I & & & \\ & & & \Delta_1 & & \\ & & & & \ddots & \\ & & & & & \Delta_r \end{bmatrix}, \quad \|\delta_i\|_\infty \le 1 \text{ or } |\delta_i| \le 1 \text{ (real)}, \ \|\Delta\|_\infty \le 1.$$

New tools are needed!

Robust performance for structured uncertainty



Recall that:

Theorem

Assume M *and* Π *are stable, and that the uncertainty set* Π *satisfies*

$$\Delta \in \Pi \implies \epsilon\Delta \in \Pi \text{ for all } \epsilon \in [0,1]$$

Then the interconnection is internally stable <u>*if and only if*</u>

$$\det(I - M\Delta)(j\omega) \neq 0,\ \forall \omega \in \mathbb{R} \cup \{\infty\},\ \forall \Delta \in \Pi.$$

When M is fixed, we want to find the "**smallest**" $\Delta \in \Pi$ which makes the matrix $I - M\Delta$ singular for some ω.

Structured singular value (SSV)

When M is fixed, we want to find the "**smallest"** $\Delta \in \Pi$ (quantified by $\bar{\sigma}(\Delta)$) which destabilizes the interconnection, or equivalently, makes the matrix $I - M\Delta$ singular for some ω.

If $\Delta \in \Pi \implies \epsilon\Delta \in \Pi$ for all complex $|\epsilon| \leq 1$, then at fixed frequency ω

$$\begin{aligned} &\inf_{\Delta \in \Pi} \{\bar{\sigma}(\Delta) : \det(I - M\Delta) = 0\} \\ &= \inf_{\Delta \in \Pi,\, \bar{\sigma}(\Delta) \leq 1} \{k : \det(I - kM\Delta) = 0\} \\ &= \frac{1}{\sup_{\Delta \in \Pi,\, \bar{\sigma}(\Delta) \leq 1} \rho(M\Delta)} \end{aligned}$$

Define

$$\mu_\Pi(M(j\omega)) := \sup_{\Delta(j\omega) \in \Pi(j\omega),\, \bar{\sigma}(\Delta(j\omega)) \leq 1} \rho(M(j\omega)\Delta(j\omega))$$

e.g., if Π represents the disk uncertainty, then $\mu_\Pi(M(j\omega)) = \bar{\sigma}(M(j\omega))$. If $\Pi = \{\delta I : |\delta| \leq 1|$, then $\mu_\Pi(M(j\omega)) = \rho(M(j\omega))$.

In general

$$\rho(M(j\omega)) \leq \mu_\Pi(M(j\omega)) \leq \bar{\sigma}(M(j\omega)).$$

Structured singular value (SSV), cont'd

If D is invertible and commutes with all $\Delta \in \Pi$, i.e., $D\Delta = \Delta D$, then

$$\mu_\Pi(M) = \mu_\Pi(DMD^{-1})$$

since $\det(I - M\Delta) = \det(I - DM\Delta D^{-1}) = \det(I - DMD^{-1}\Delta)$. Thus

$$\mu_\Pi(M) = \min_{D \in \mathscr{D}_\Pi} \mu_\Pi(DMD^{-1}) \leq \min_{D \in \mathscr{D}_\Pi} \bar{\sigma}(DMD^{-1})$$

with

$$\mathscr{D}_\Pi = \{D : D\Delta = \Delta D, \forall \Delta \in \Pi\}.$$

In particular, the inequality becomes equality, i.e.,

$$\mu_\Pi(M) = \min_{D \in \mathscr{D}_\Pi} \bar{\sigma}(DMD^{-1})$$

whenever Δ has the following structure

$$\Delta = \operatorname{diag}(\delta_1 I_{r_1}, \cdots, \delta_S I_{r_S}, \Delta_1, \cdots, \Delta_F) \text{ with } \delta_i \in \mathbb{C},\ \Delta_j \in \mathbb{C}^{m_j \times m_j}$$

with $2S + F \leq 3$.

Structured singular value (SSV), cont'd

If Δ is a full block complex uncertainty, then $\Delta D = D\Delta$ iff D is diagonal. Hence for

$$\Delta = \mathrm{diag}(\delta_1 I_{r_1}, \cdots, \delta_S I_{r_S}, \Delta_1, \cdots, \Delta_F) \text{ with } \delta_i \in \mathbb{C},\ \Delta_j \in \mathbb{C}^{m_j \times m_j}$$

Take

$$D = \mathrm{diag}(d_1, \cdots, d_S, \tilde{d}_1 I, \cdots, \tilde{d}_F I).$$

SSV example

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \in RH_\infty,\ \|\Delta\|_\infty \le 1 \implies \mu(N) = \min_{D\Delta=\Delta D,\ D, D^{-1} \in H_\infty} \bar{\sigma}(DND^{-1})$$

Consider

$$D_\omega = \begin{bmatrix} d_1(\omega)I & 0 \\ 0 & d_2(\omega)I \end{bmatrix}$$

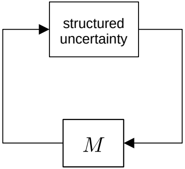
$$\implies \mu(N) = \min_{d_1(\omega), d_2(\omega)} \bar{\sigma}\left(\begin{bmatrix} N_{11} & \frac{d_1(\omega)}{d_2(\omega)} N_{12} \\ \frac{d_2(\omega)}{d_1(\omega)} N_{21} & N_{22} \end{bmatrix}\right) = \min_{d(\omega)} \bar{\sigma}\left(\begin{bmatrix} N_{11} & d(\omega) N_{12} \\ \frac{1}{d(\omega)} N_{21} & N_{22} \end{bmatrix}\right)$$

Special case:

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix},\ \delta_1, \delta_2 \in \mathbb{C},\ |\delta_1|, |\delta_2| \le 1, \quad N = \begin{bmatrix} a & a \\ b & b \end{bmatrix} \in \mathbb{C}^{2\times 2}$$

$$\mu(N) = \min_{d(\omega)} \bar{\sigma}\left(\begin{bmatrix} a & d(\omega)a \\ \frac{1}{d(\omega)} b & b \end{bmatrix}\right) = \min_{d(\omega)} \sqrt{|a|^2 + |da|^2 + \left|\frac{b}{d}\right|^2 + |b|^2} = |a| + |b|$$

Robust performance for structured uncertainty



Theorem

Assume M and Π are stable, and that the uncertainty set Π satisfies

$$\Delta \in \Pi \implies \epsilon\Delta \in \Pi \text{ for all } \epsilon \in [0,1]$$

Then the interconnection is internally stable <u>if and only if</u>

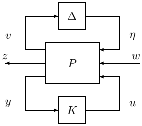
$$\det(I - M\Delta)(j\omega) \neq 0, \ \forall \omega \in \mathbb{R} \cup \{\infty\}, \ \forall \Delta \in \Pi.$$

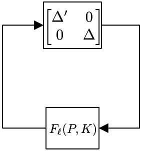
Theorem

Let M be stable and Π a complex stable structured uncertainty. Then the interconnection is internally stable for all $\Delta \in \Pi$ with $\bar{\sigma}(\Delta) \leq 1$ if and only if

$$\mu_{\Pi}(M(j\omega)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}$$

Robust performance for structured uncertainty





robust stability

$$\forall \Delta'' = \begin{bmatrix} \Delta' & 0 \\ 0 & \Delta \end{bmatrix} \qquad \Pi' = \left\{ \begin{bmatrix} \Delta' & 0 \\ 0 & \Delta \end{bmatrix} : \Delta \in \Pi,\ \Delta' \in \mathbb{C}^{r \times s} \right\}$$

$$\|\Delta'\|_\infty \leq 1$$

Now the robust performance condition reads

$$\mu_\Delta(F_\ell(P,K)_{1,1}) < 1 \text{ (robust stability)}$$

$$\mu_{\Delta''}(F_\ell(P,K)) < 1 \text{ (robust performance)}$$

It is easy to verify that the second condition implies the first one!

Theorem

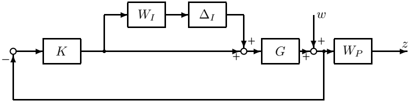
Suppose that K (nominally) stabilizes P_{33}. Let Π be a complex stable uncertainty. Then the interconnection is internally stable and satisfies the robust performance condition

$$\|F_u(F_\ell(P,K),\Delta)\|_\infty \leq 1$$

for all $\Delta \in \Pi$ with $\bar{\sigma}(\Delta) \leq 1$ if and only if

$$\mu_{\Pi'}(F_\ell(P,K)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

Example



Assumption: All signals and transfer functions are scalar-valued. Δ_I is complex, with $\|\Delta_I\|_\infty \leq 1$.

Robust performance:

$$\|T_{zw}\|_\infty \leq 1, \quad \forall \Delta_I.$$

$$F_\ell\left(\begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -1 & -G \end{bmatrix}, K\right) = \begin{bmatrix} -W_I T & -W_I K S \\ W_P G S & W_P S \end{bmatrix}$$

$$\mu\left(\begin{bmatrix} -W_I T & -W_I K S \\ W_P G S & W_P S \end{bmatrix}\right) = \mu\left(\begin{bmatrix} -W_I T & -W_I K G S \\ W_P S & W_P S \end{bmatrix}\right) \quad \left(\mu(D \cdot D^{-1}),\ D = \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \mu\left(\begin{bmatrix} -W_I T & -W_I T \\ W_P S & W_P S \end{bmatrix}\right)$$

$$= |W_I T| + |W_P S|.$$

$\Longrightarrow$ robust performance condition: $|W_I T| + |W_P S| < 1$.

Robust performance for structured uncertainty

Corollary

Suppose that K (nominally) stabilizes P_{33}. Let Π be a complex stable uncertainty. Then the interconnection is internally stable and satisfies the robust performance condition

$$\|F_u(F_\ell(P,K),\Delta)\|_\infty \leq \beta$$

for all $\Delta \in \Pi$ with $\bar{\sigma}(\Delta) \leq \frac{1}{\beta}$ if and only if

$$\mu_{\Pi'}(F_\ell(P,K)) < \beta, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

μ-synthesis and DK-iteration

Goal: given complex uncertainty description $\Delta \in \Pi$, find K such that

$$\mu_{\Pi'}(F_\ell(P,K)) < 1.$$

Recall

$$\mu_{\Pi'}(M(j\omega)) \le \min_{D \in \mathscr{D}_{\Pi'}} \bar{\sigma}(DMD^{-1}(j\omega))$$

Compute upper bound

$$\min_K \min_{D \in \mathscr{D}_\Pi, D, D^{-1} \in H_\infty} \|DF_\ell(P,K)D^{-1}\|_\infty$$

- K**-step**: fix $D(s)$, solve (H_∞-synthesis)

$$\min_K \|DF_\ell(P,K)D^{-1}\|_\infty$$

- D**-step**: fix $K(s)$, solve frequency-wise (convex program)

$$\min_{D \in \mathscr{D}_\Pi, D, D^{-1} \in H_\infty} \bar{\sigma}(DF_\ell(P,K)D^{-1}(j\omega))$$

The K-step

$$\min_K \|DF_\ell(P,K)D^{-1}\|_\infty$$

$$DF_\ell(P,K)D^{-1} = F_\ell\left(\begin{bmatrix} D & \\ & I \end{bmatrix} P \begin{bmatrix} D & \\ & I \end{bmatrix}^{-1}, K\right)$$

For problem

$$\min_K \|F_\ell(G,K)\|_\infty$$

Matlab code:

```
[K, CL, gamma, info] = hinfsyn(G,input_num,output_num);
```

The D-step

$$\min_{D \in \mathscr{D}_\Pi, D, D^{-1} \in H_\infty} \bar{\sigma}(D F_\ell(P,K) D^{-1}(j\omega))$$

- Minimize to get D_ω across frequency.
- Find $D(s), D^{-1}(s) \in H_\infty$ such that $|D(j\omega)| \approx D_\omega$ across frequency (interpolation theory Youla & Saito 1967).

Example, consider the two-block uncertainty

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \in RH_\infty, \ \|\Delta\|_\infty \leq 1$$

Then

$$\mu(N) = \min_{D\Delta = \Delta D, \ D, D^{-1} \in H_\infty} \bar{\sigma}(DND^{-1})$$

Matlab code:

```
N = frd(lft(P,K),omega);
[mu_bnds, mu_info] = mussv(N, blk);
[~, VSigma, ~] = mussvextract(mu_info);
D = VSigma.DLeft;
d1 = fitfrd(genphase(D(1,1)),4) ...
```

Example on μ-synthesis

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}, \quad W_I(s) = \frac{s+0.2}{0.5s+1}, \quad W_P(s) = \frac{0.5s+0.05}{s}$$

$$\Delta_I : \text{two } 1\times 1 \text{ blocks}, \quad \Delta_P : \text{ one } 2\times 2 \text{ block}, \ \Delta = \begin{bmatrix} \Delta_I & \\ & \Delta_P \end{bmatrix} \in \mathbb{C}^{4\times 4}$$

$$D = \text{diag}\{d_1, d_2, I_2\} \implies \text{blk = [1,1; 1,1; 2,2]}$$