Robust Control

Lecture 2

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Review of last lecture

Dynamic (frequency-dependent) uncertainties.

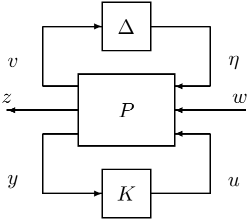
- Unmodeled dynamics at high frequency (phase completely unknown at high frequencies!)
- Imperfect measurements $\Rightarrow$ uncertain inputs.
- Nonlinearities.

Parametric uncertainties.

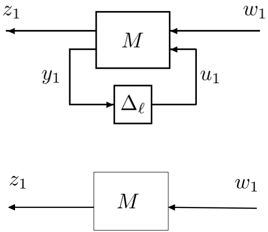
- Inaccurate description of components.
- Variations of system parameters.

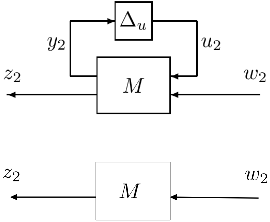
Review of last lecture

General framework:



ℓLFT and uLFT:





Review of last lecture

G <u>proper and stable</u>. H_∞-norm of G:

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \{\text{largest signular value of } G(j\omega)\}$$
$$=: \overline{\sigma}(G(j\omega))$$

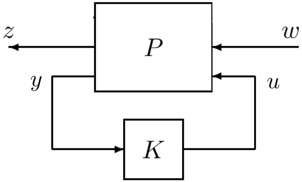
Definition

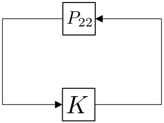
The H_∞ *space* of transfer matrices consists of all matrix-valued functions that are:

- **Stable**, i.e., analytic in the open RHP;
- **Bounded** (in H_∞-norm) in the open RHP.

The subspace of real rational H_∞ functions is denoted by RH_∞

Review of last lecture





The ℓLFT is well-posed iff

$$\begin{bmatrix} I & -K(j\infty) \\ -P_{22}(j\infty) & I \end{bmatrix} \text{ is invertible}$$

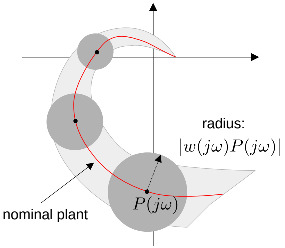
The ℓLFT is internally stable iff it is well-posed, and

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} \text{ is stable.}$$

This lecture

- Modeling of uncertainties
- Nominal and robust performance specifications

Multiplicative uncertainties: SISO



$$|\tilde{P}(j\omega) - P(j\omega)| \leq |w(j\omega)P(j\omega)|, \quad \forall \omega \in \mathbb{R}$$

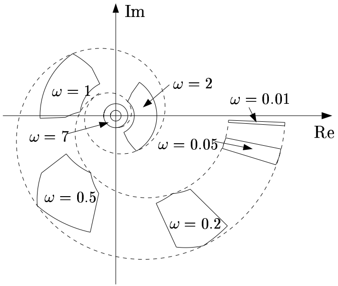
$$\left|\frac{\tilde{P} - P}{wP}\right|(j\omega) \leq 1, \quad \forall \omega \in \mathbb{R} \quad \Leftrightarrow \quad \left\|\frac{\tilde{P} - P}{wP}\right\|_\infty \leq 1$$

Let $\Delta = \frac{\tilde{P}-P}{wP}$, then

$$\tilde{P} = (1 + w\Delta)P, \quad \|\Delta\|_\infty \leq 1$$

Example

$$\tilde{P}(s) = \frac{ke^{-\theta s}}{\tau s+1}, \quad 2 \leq k,\theta,\tau \leq 3$$

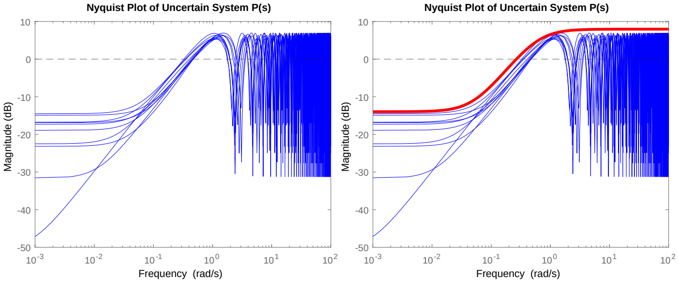


How to choose a nominal model (SISO):

- Simple, e.g., low-order, delay free.
- A model of mean parameter values.
- The central plant obtained from the Nyquist plot.

Example, cont'd

Choose $w(s)$ so that $\left|\frac{\tilde{P}-P}{P}\right|(j\omega) \le |w(j\omega)|, \quad \forall\omega.$



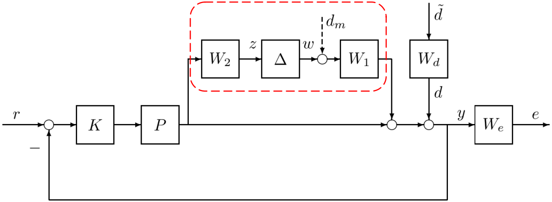
- Candidate

$$w(s) = \frac{\tau s + \text{low freq. gain}}{(\tau/\text{high freq. gain}) \times s + 1}, \quad 1/\tau: \text{freq. with 100\% uncertainty}$$

- Gains at low and high frequency: 0.2, 2.33, and $1/\tau \approx 0.25$, or $\tau \approx 4$.

$$w(s) = \frac{4s + 0.2}{(4/2.33)s + 1}.$$

Multiplicative uncertainties: MIMO



$$\tilde{P} = (I + W_1 \Delta W_2) P, \quad \|\Delta\|_\infty \leq 1$$

$$\tilde{P} = P(I + W_1 \Delta W_2), \quad \|\Delta\|_\infty \leq 1$$

Parametric uncertainties

Gain uncertainty:

$$\tilde{P} = kP_0(s); \quad k_{\min} \le k \le k_{max}$$

$$\tilde{P} = \underbrace{\overline{k}P_0(s)}_{P(s)}(1 + \underbrace{w}_{\text{constant}} \Delta), \quad |\Delta| \le 1, \text{ (\textit{not} } H_\infty\text{-norm!)}$$

where w, $\bar{k}$ are constants.

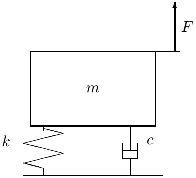
Pole uncertainty:

$$\tilde{P} = \frac{1}{s-p} P_0(s), \quad p_{\min} \le p \le p_{\max}$$

$$\tilde{P} = \frac{P(s)}{1 + w(s)\Delta(s)}, \quad |\Delta| \le 1$$

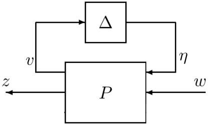
where $w(s) = \frac{r_a \overline{a}}{s - \overline{a}}$.

Example



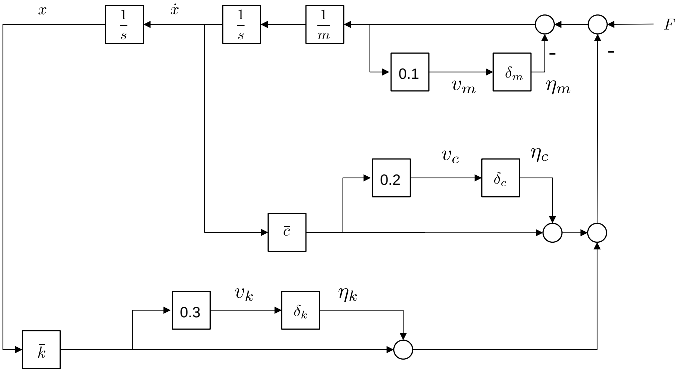
$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}$$

	nominal value	error	actual mass	normalized error
m	$\bar{m}$	$\pm 10\%$	$\bar{m}(1+0.1\delta_m)$	$\lvert\delta_m\rvert \le 1$
c	$\bar{c}$	$\pm 20\%$	$\bar{c}(1+0.2\delta_c)$	$\lvert\delta_c\rvert \le 1$
k	$\bar{k}$	$\pm 30\%$	$\bar{k}(1+0.3\delta_k)$	$\lvert\delta_k\rvert \le 1$



Choose $v = [v_m,\ v_c,\ v_k]^\top$, $\eta = [\eta_m,\ \eta_c,\ \eta_k]^\top$.

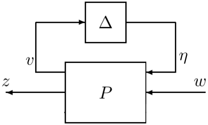
Example, cont'd



Find P, s.t.

$$\begin{bmatrix} v \\ z \end{bmatrix} = P \begin{bmatrix} \eta \\ w \end{bmatrix}$$

Example, cont'd



Exogenous input $w = F$, exogenous output: $z = x$.

$$\begin{bmatrix} v_m \\ v_c \\ v_k \\ x \end{bmatrix} = \underbrace{\frac{1}{\bar{m}s^2 + \bar{c}s + \bar{k}} \begin{bmatrix} -0.1\bar{m}s^2 & -0.1\bar{m}s^2 & -0.1\bar{m}s^2 & 0.1\bar{m}s^2 \\ -0.2\bar{c}s & -0.2\bar{c}s & -0.2\bar{c}s & 0.2\bar{c}s \\ -0.3\bar{k} & -0.3\bar{k} & -0.3\bar{k} & 0.3\bar{k} \\ -1 & -1 & -1 & 1 \end{bmatrix}}_{P} \begin{bmatrix} \eta_m \\ \eta_c \\ \eta_k \\ F \end{bmatrix}$$

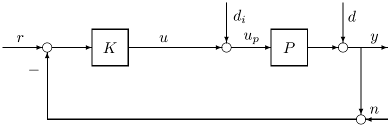
Hence

$$x = F_u(P, \Delta)F, \quad \Delta = \begin{bmatrix} \delta_m & & \\ & \delta_c & \\ & & \delta_k \end{bmatrix}$$

with

$$|\delta_m|, \ |\delta_c|, \ |\delta_k| \leq 1.$$

Nominal Performance specification



input, output loop transfer matrix	$L_i = KP$, $L_o = PK$
input, output sensitivity matrix	$S_i = (I + L_i)^{-1}$, $S_o = (I + L_o)^{-1}$
input, output complementary sensitivity matrix	$T_i = I - S_i$, $T_o = I - S_o$

Disturbance rejection

$$y = T_o(r-n) + S_o P d_i + S_o d \text{ (plant output)}$$
$$u_p = KS_o(r-n) - KS_o d + S_i d_i \text{ (plant input)}$$
$$u = KS_o(r-n) - KS_o d - T_i d_i \text{ (controller output)}$$
$$r - y = S_o(r-d) + T_o n - S_o P d_i \text{ (reference error)}$$

	signal	make small	design
disturbance rejection	y	$\bar{\sigma}(S_o)$, $\bar{\sigma}(S_o P)$	$\underline{\sigma}(L_o) \gg 1$ (for d), $\underline{\sigma}(K) \gg 1$ (for d_i)
	u_p	$\bar{\sigma}(KS_o)$, $\bar{\sigma}(S_i)$	$\underline{\sigma}(L_i) \gg 1$ (for d_i), $\underline{\sigma}(P) \gg 1$ (for d)

$$\bar{\sigma}(S_o) = 1/\underline{\sigma}(I + PK), \quad \bar{\sigma}(S_o P) = \bar{\sigma}((I + PK)^{-1} P)$$
$$\bar{\sigma}(PK) \gg 1 \;\&\; \bar{\sigma}(S_o P) \approx \bar{\sigma}(K^{-1}) = 1/\underline{\sigma}(K) \ll 1$$

$$\bar{\sigma}(S_i) = 1/\underline{\sigma}(I + KP), \quad \bar{\sigma}(KS_o) = \bar{\sigma}((I + KP)^{-1} K)$$

$$\bar{\sigma}(KP) \gg 1 \;\&\; \bar{\sigma}(KS_o) \approx \bar{\sigma}(P^{-1}) = 1/\underline{\sigma}(P) \ll 1$$

Achieve high loop and controller gain in the necessary frequency range.

Conflicting objectives

- For

$$y = T_o(r-n) + S_o P d_i + S_o d \text{ (plant output)}$$

when S_o is small (or L_o big), then

$$y \approx r - n \implies \text{ measurement noise passes through}$$

- When L_o or L_i large, then outside the bandwidth of P (i.e., P small)

$$K \gg 1 \implies \text{ controller saturation}$$

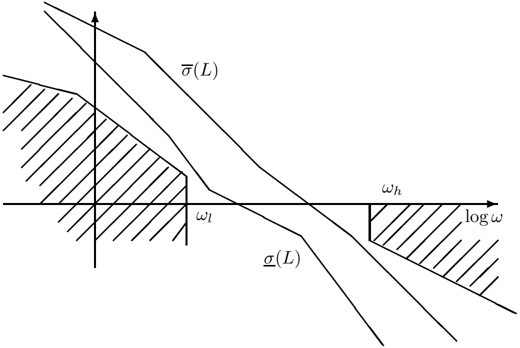
At low frequency (disturbance rejection, reference tracking etc.)

$$\underline{\sigma}(PK) \gg 1,\ \underline{\sigma}(KP) \gg 1,\ \underline{\sigma}(K) \gg 1$$

At high frequency (sensor noise rejection, etc.)

$$\bar{\sigma}(PK) \ll 1,\ \bar{\sigma}(KP) \ll 1,\ \bar{\sigma}(K) \ll M$$

Conflicting objectives



Weighted H_∞ performance

$$\text{track., disturb. attenu.:} \quad \begin{array}{ll} |S(j\omega)| & \leq \epsilon, \quad \forall \omega \leq \omega_0 \\ |S(j\omega)| & \leq M, \quad \forall \omega > \omega_0 \end{array}$$

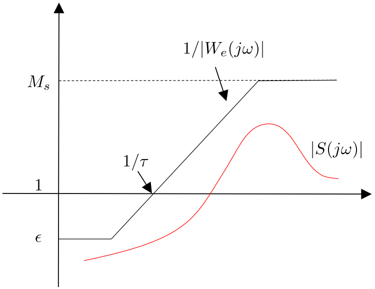
Let

$$|W_e(j\omega)| = \begin{cases} 1/\epsilon, & \forall \omega \leq \omega_0 \\ 1/M, & \forall \omega > \omega_0 \end{cases}$$

Equivalently

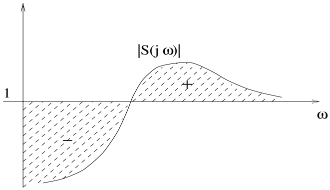
$$|W_e(j\omega)S(j\omega)| \leq 1, \quad \forall \omega \quad \Leftrightarrow \quad \|W_e S\|_\infty \leq 1.$$

A finer weighting function:



$$\frac{1}{W_e(s)} = \frac{\tau s + \epsilon}{(\tau/M_s)s + 1}$$

Bode's sensitivity integral



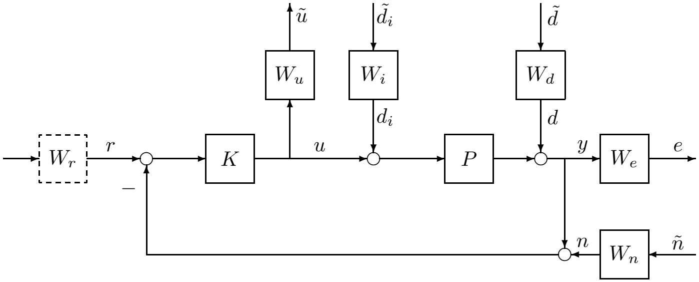
$$\int_0^\infty \ln|S(j\omega)|\mathrm{d}\omega = 0$$

when L has no open RHP zeros and its relative degree ≥ 2.

Selection of weighting functions

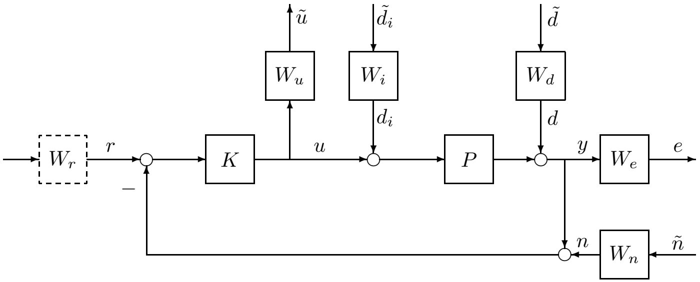
- Non-trivial, no general formulas exist
- Requires ad hoc fixing and fine tuning

Weighted H_∞ performance



- W_d, W_i, W_n: reflect the frequency contents of the disturbances / noise d, d_i, n.
- W_e: reflect the shape of certain closed-loop transfer matrices.
- W_u: restrictions on the control or actuator signals.
- W_r: shape of the command/reference.

Example



Assume $d_i = 0$, $n = 0$, try to analyze the “worst case” impact of $\tilde{d}$ on $(e, \tilde{u})$ (equiv. L_2-gain):

$$\sup_{\|\tilde{d}\|_2 \leq 1} \{\|e(\cdot)\|_2^2 + c^2 \|\tilde{u}(\cdot)\|_2^2\} = \left\| \begin{bmatrix} W_e S_o W_d \\ c W_u K S_o W_d \end{bmatrix} \right\|_\infty^2 \text{ (by def.)}$$

Analytic constraints

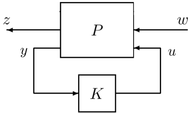
Let p and z be the open RHP poles and zeros of L. Suppose that the closed-loop system is stable, then

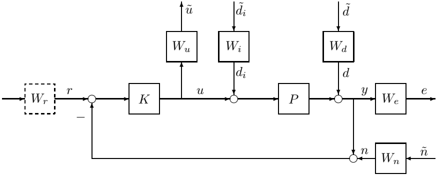
$$S(p) = 0,\ T(p) = 1$$
$$S(z) = 1,\ T(z) = 0$$

Hence

$$\|W_e S\|_\infty \geq |W_e(z)| \text{ if } \exists \text{ open RHP zero } z$$

Synthesis problem: design K





Assume $d_i = 0$, $n = 0$, design K that minimizes the "worst case" impact of $\tilde{d}$ on $(e, \tilde{u})$ (equiv. L_2-gain):

$$\min_K \left\| \begin{bmatrix} W_e S_o W_d \\ c W_u K S_o W_d \end{bmatrix} \right\|_\infty^2,$$

or find K such that

$$\left\| \begin{bmatrix} W_e S_o W_d \\ c W_u K S_o W_d \end{bmatrix} \right\|_\infty^2 \leq \rho$$

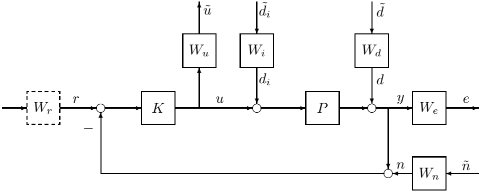
for some given level ρ. Choose $w = \tilde{d}$, $z = \mathrm{col}(e, \tilde{u})$,

$$\begin{bmatrix} W_e S_o W_d \\ c W_u K S_o W_d \end{bmatrix} = F_\ell(M, K)$$

for a generalized plant M. Thus the synthesis problem can be written compactly as

$$\min_K F_\ell(M,K), \text{ or find } K \text{ s.t. } \|F_\ell(M,K)\|_\infty \leq 1.$$

Class exercise



Write $\|W_e S_o W_d\|_\infty \leq 1$ in the form of

$$\|F_\ell(M, K)\|_\infty \leq 1$$