Robust Control

Lecture 1

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Course Information

- ~ 7 Lectures (Dongjun and Richard), ~ 7 exercises (Dongjun)
- Textbooks: Essentials of Robust Control etc.
- Schedule and material: see course page
- Examination: Exercises + Handins + Exam
- Collaboration encouraged on exercises and handins!
- Handins are due before the exercise session, email to: dongjun.wu@control.lth.se with subject *Robust control handin X*
- Prepare so that you are able to share your solutions to the exercises at the session. (Take a photo of handwritten notes or typeset)

Syllabus

Lecture 1, [Zhou 9,3,4,5]

Abstract uncertainty, LFTs, well-posedness, internal stability, review of LTI.

Lecture 2, [Zhou 11,6, 8]

Uncertainty representation and performance specifications

Lecture 3, [Zhou 10]

μ-synthesis

Lecture 4, [Zhou 14, 12, ...]

H_∞ synthesis, AREs

Lecture 5, [...]

H_∞-loop shaping

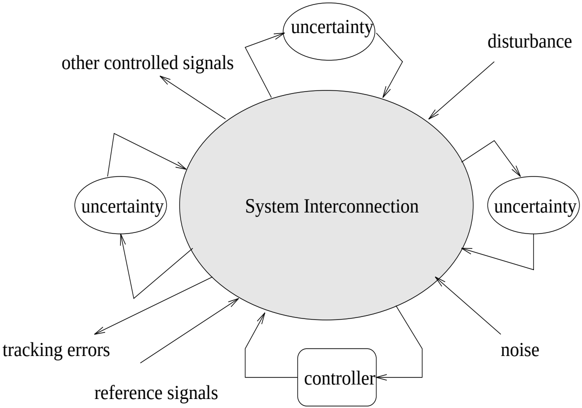
Lecture 6

ν-gap, IQC.

Why robust control?

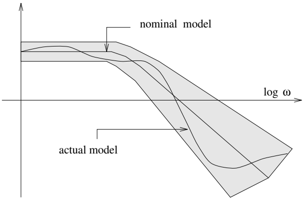
Uncertainty is the natural habitat of human life — though the hope of escaping uncertainty is the engine of human life pursuits.

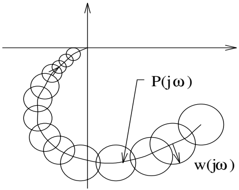
— *Bauman, Zygmunt. The art of life. John Wiley & Sons, 2013.*



Uncertainties

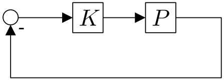
- **Dynamic (frequency-dependent) uncertainties.**
  - Unmodeled dynamics at high frequency (phase completely unknown at high frequencies!)
  - Imperfect measurements ⇒ uncertain inputs.
  - Nonlinearities.
- **Parametric uncertainties.**
  - Inaccurate description of components.
  - Variations of system parameters.



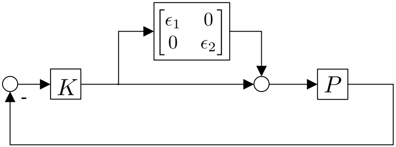


Example (Doyle, 1986): spinning satelite

$$P(s) = \frac{1}{s^2+a^2}\begin{bmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{bmatrix}, \quad a = 10.$$



Controller: $K = I$. Closed-loop poles: $\{-1, -1\} \Rightarrow$ stable.



Take $\epsilon_1 = -\epsilon_2 = 0.11$. Closed-loop poles: $\{-2.1,\ 0.1\} \Rightarrow$ unstable.

Essentials of classic control

- **Stability**: Cope with unknown initial conditions & perturbations
- **High-gain** at *low frequencies*: Achieve tracking & reject disturbances
- **Low-gain** at *high frequency*: Reduce effect of sensor noise and large plant-model mismatch
- **Gain- and phase-margins**: Render stability and performance robust against (possibly large) plant-model mismatch
- **Tool**: Manual shaping of loop transfer function

Modern Robust Control: Set of tools for systematically coping with all these issues for complex interconnections and by directly imposing desired properties on the controlled system.

Workflow

Choose a nominal model (how?)

$\Downarrow$

Quantify uncertainties w.r.t. the nominal model (how?)

$\Downarrow$

Specify desired robust performance

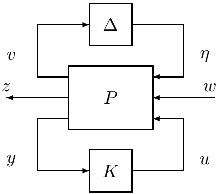
$\Downarrow$

Solve the robust control problem

$\Downarrow$

Validation

General framework



$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) & P_{13}(s) \\ P_{21}(s) & P_{22}(s) & P_{23}(s) \\ P_{31}(s) & P_{32}(s) & P_{33}(s) \end{bmatrix} \begin{bmatrix} \eta \\ w \\ u \end{bmatrix},$$

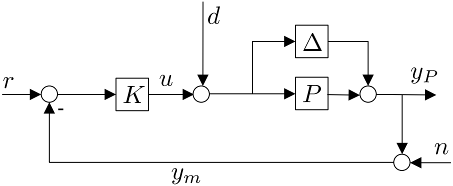
$$\eta = \Delta v$$

$$u = Ky$$

P	nominal (generalized) plant
Δ	all uncertainties
K	controller
w	exogenous inputs (reference, disturbance, etc.)
z	exogenous output (controlled variable)
y	controller input
u	controller output

Example: 1DOF system

Write the following system in standard form. The plant is subject to *additive uncertainty*: $P+\Delta$.



$$w = \begin{bmatrix} d \\ r \\ n \end{bmatrix}, \qquad \begin{aligned} v &= u + d \\ z &= P(u+d) + \eta \\ y &= r - n - P(u+d) - \eta \end{aligned}$$

$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = \underbrace{\left[\begin{array}{c|ccc|c} 0 & I & 0 & 0 & I \\ I & P & -I & 0 & P \\ -I & -P & I & -I & -P \end{array}\right]}_{\text{generalized plant}} \begin{bmatrix} \eta \\ w \\ u \end{bmatrix}$$

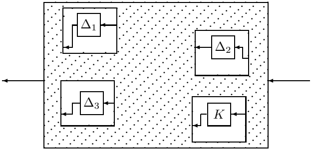
$$u = Ky$$

$$\eta = \Delta v$$

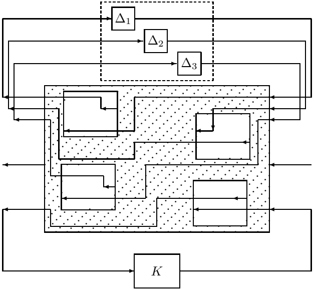
Other types of uncertainties

$P + W_1 \Delta W_2$	additive uncertainty
$P(I + W_1 \Delta W_2)$	input uncertainty
$(I + W_1 \Delta W_2)P$	output uncertainty
$P(I + W_1 \Delta W_2)^{-1}$	
$(I + W_1 \Delta W_2)^{-1} P$	
$P(I + W_1 \Delta W_2 P)^{-1}$	
$(R + \Delta_R)^{-1}(S + \Delta_S)$ coprime factorization: $P = R^{-1} S$	coprime factor uncertainty

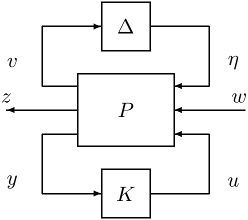
Pulling out uncertainties



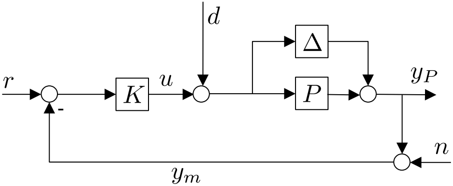
$\Rightarrow$

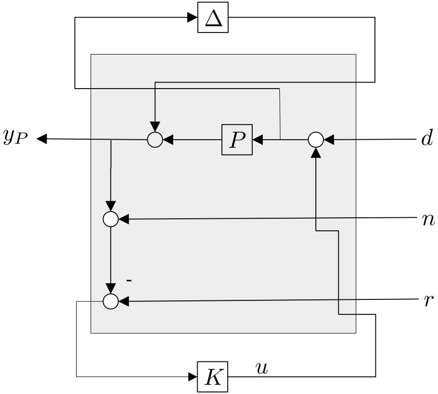


$\Rightarrow$



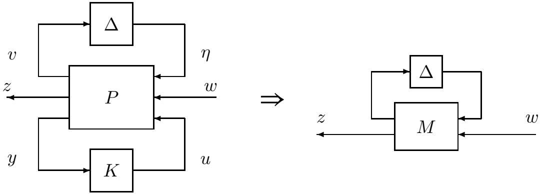
Example: 1DOF system





The $M\Delta$-structure (rob. perf. analysis)

When K is fixed:



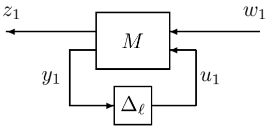
$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \eta \\ w \\ u \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \eta \\ w \\ Ky \end{bmatrix}$$

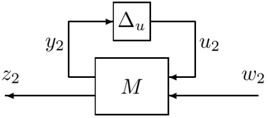
$$y = [P_{31}\ P_{32}] \begin{bmatrix} \eta \\ w \end{bmatrix} + P_{33} K y \quad \Rightarrow \quad y = (I - P_{33}K)^{-1} [P_{31}\ P_{32}] \begin{bmatrix} \eta \\ w \end{bmatrix}$$

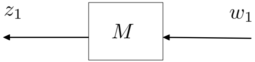
$$\begin{bmatrix} v \\ z \end{bmatrix} = \underbrace{\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix} K (I - P_{33}K)^{-1} [P_{31}\ P_{32}] \right)}_{M := F_\ell(P,K)} \begin{bmatrix} \eta \\ w \end{bmatrix}$$

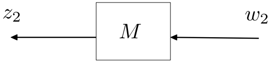
Linear fractional transformation (LFT)

Compute the transfer functions from w_i to z_i:









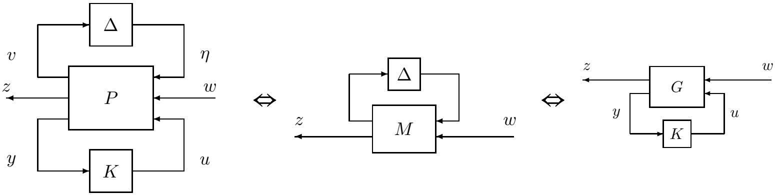
$$z_1 = \underbrace{[M_{11} + M_{12}\Delta_\ell(I - M_{22}\Delta_\ell)^{-1}M_{21}]}_{F_\ell(M,\Delta_\ell)} w_1$$

$F_\ell(M, \Delta_\ell)$: lower LFT (ℓLFT) of M w.r.t Δ_ℓ.

$$z_2 = \underbrace{[M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}]}_{F_u(M,\Delta_u)} w_2$$

$F_u(M, \Delta_u)$: upper LFT (uLFT) of M w.r.t Δ_ℓ.

Revisit of the general framework



$$M = F_\ell(P, K)$$

$$G = F_u(P, \Delta)$$

$$z = F_u(F_\ell(P,K), \Delta)\, w$$
$$= F_\ell(F_u(P,\Delta), K)\, w$$

Caveat

ℓLFT: $F_\ell(M,\Delta_\ell) = M_{11} + M_{12}\Delta_\ell(I - M_{22}\Delta_\ell)^{-1}M_{21}$

defined only when $I - M_{22}\Delta_\ell$ is *invertible*!

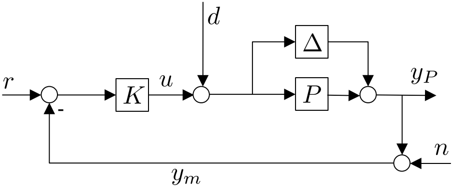
uLFT: $F_u(M,\Delta_\ell) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$

defined only when $I - M_{11}\Delta_u$ is *invertible*!

(Come back at well-posedness)

Example

Compute the transfer matrix from $\begin{bmatrix} d \\ r \\ n \end{bmatrix}$ to y_P using LFT.



Short summary

What we have learned:

- The origins of uncertainties (dynamical + parametric)
- How to represent (abstract) uncertainties in a unified framework
- Linear fractional transformations

Questions remain:

- How to choose Δ (dynamical + parametric)?
  - Which classes of transfer matrices are we interested in? (crucial to the solvability of the robust control problem)
  - What structures of Δ do we allow?
- How to formulate the robust control problem?

State-space & I/O

System in *state-space* form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Input-output representation (transfer matrix): $y(s) = G(s)u(s)$

$$G(s) = C(sI - A)^{-1}B + D$$
$$=: G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad (\text{not } \begin{bmatrix} A & B \\ C & D \end{bmatrix}!)$$

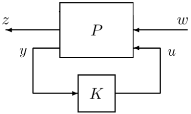
We call the state space model (A, B, C, D) a *realization of G.*

Some algebra: $G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$, $G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$

$$G_1 G_2 = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right], \quad G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Exercise: write the state space realization of $G^\top(s)$.

State space realization of LFT



State space realization of:

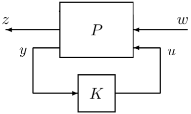
$$\dot{x} = Ax + B_1 w + B_2 u \qquad \dot{x}_K = A_K x_K + B_K y$$
$$z = C_1 x + D_{11} w + D_{12} u \qquad u = C_K x_K + D_K y$$
$$y = C_2 x + D_{12} w + D_{22} u$$

$$\dot{x}_K = A_K x_k + B_K (C_2 x + D_{12} w + D_{22} u)$$
$$u = C_K x_K + D_K C_2 x + D_K D_{12} w + D_K D_{22} u$$
$$(I - D_K D_{22}) u = C_K x_K + D_K C_2 x + D_K D_{12} w$$

Definition

The ℓLFT is well-posed if $I - D_K D_{22}$, (or equivalently $I - D_{22} D_K$, $I - P_{22}(j\infty) K(j\infty)$) is invertible.

State space realization of LFT



State space realization of:

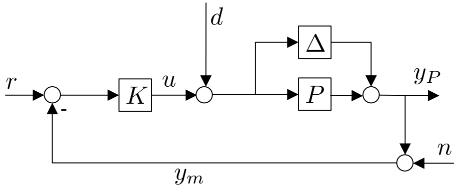
$$\dot{x} = Ax + B_1 w + B_2 u \qquad \dot{x}_K = A_K x_K + B_K y$$
$$z = C_1 x + D_{11} w + D_{12} u \qquad u = C_K x_K + D_K y$$
$$y = C_2 x + D_{21} w + D_{22} u$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w$$

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w$$

The ℓLFT is well-posed if and only if $\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} = \begin{bmatrix} I & -K(j\infty) \\ -P_{22}(j\infty) & I \end{bmatrix}$ is invertible.

Example



$$\begin{bmatrix} v \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|ccc|c} 0 & I & 0 & 0 & I \\ I & P & -I & 0 & P \\ -I & -P & I & -I & -P \end{array}\right] \begin{bmatrix} \eta \\ w \\ u \end{bmatrix}, \qquad \begin{aligned} \eta &= \Delta v \\ u &= Ky \end{aligned}$$

Set $\Delta = 0$, then $\eta = 0$, and

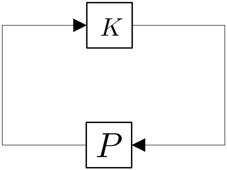
$$\begin{bmatrix} z \\ y \end{bmatrix} = \left[\begin{array}{ccc|c} P & -I & 0 & P \\ -P & I & -I & -P \end{array}\right] \begin{bmatrix} w \\ u \end{bmatrix}, \qquad u = Ky$$

Thus according to Definition, the system is well-posed if and only if

$$I - (-P(j\infty))K(j\infty) = I + P(j\infty)K(j\infty) \text{ is invertible.}$$

Class exercise

Show that the connection is well-posed if and only if $I - P(j\infty)K(j\infty)$ is invertible:



Controllability and Observability

- (A, B) is *controllable (stabilizable)* if

$$\text{rank}[\lambda I - A \ \ B] = n$$

  for all $\lambda \in \mathbb{C}$ ($\lambda \in$ closed RHP);
- (C, A) is *observable* (*detectable*) if

$$\text{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$$

  has full column rank for all $\lambda \in \mathbb{C}$ ($\lambda \in$ closed RHP).

We say that a state space realization (A, B, C, D) for $G(s)$ is *minimum* if (A, B) is controllable and (C, A) observable.

Poles and zeros

Definition

Let $G(s)$ be a transfer matrix.

- $p \in \mathbb{C}$ is a *pole* if it is a pole of an entry of $G(s)$.
- $z \in \mathbb{C}$ is a *(transmission) zero* if $G(z)$ loses rank.
- G is *stable* if every entry of G is stable.
- G is *proper* if $G(j\infty)$ is finite; *strictly proper* if $G(j\infty) = 0$.
- G is *minimum phase* if it does *not* have RHP zeros or time delays; otherwise it is *non-minimum phase.*

Convention

All P (plant) and K (controller) in this course are assumed to be proper.

H_∞-norm

G proper and stable. H_∞-norm of G:

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \{\text{largest signular value of } G(j\omega)\}$$
$$=: \overline{\sigma}(G(j\omega))$$

Scalar case: $\|g\|_\infty = \sup_\omega |g(j\omega)|$.

Equivalently (when defined):

$$\|G\|_\infty = \sup_{u \in L_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\text{pow}(u) \leq 1} \frac{\text{pow}(Gu)}{\text{pow}(u)}$$

where $\text{pow}(u) := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|^2 \mathrm{d}t\right)^{1/2}$.

$\|G\|_\infty$ is the L_2*-gain* of the system, and the *largest possible amplification of asymptotic signal power/energy*.

H_∞-space

Definition

The H_∞ *space* of transfer matrices consists of all matrix-valued functions that are

- **Stable**, i.e., analytic in the open RHP;
- **Bounded** (in H_∞-norm) in the open RHP.

The subspace of real rational H_∞ functions is denoted by RH_∞

Implication: for G real rational

$$G \in H_\infty \quad \Leftrightarrow \quad G \text{ stable and proper}$$

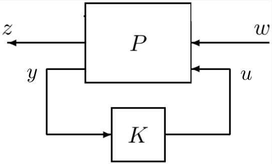
Note that (by the maximum modulus theorem)

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(G(j\omega)) = \sup_{\mathrm{Re}(s) \geq 0} \overline{\sigma}(G(s)) = \sup_{\mathrm{Re}(s) > 0} \overline{\sigma}(G(s))$$

Internal stability of LFT

Definition

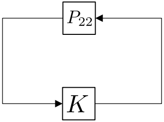
Let P, K be transfer matrices. The LFT



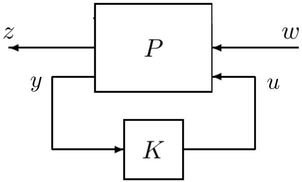
is well-posed. Let (x_P, x_K) be its internal state. Then it is *stable* if $w = 0$ implies

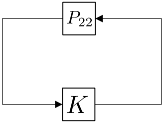
$$(x_P(t), x_K(t)) \to 0 \text{ as } t \to \infty$$

for any initial condition $(x_P(0), x_K(0))$.



Internal stability of LFT, cont'd





$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w$$

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w$$

The ℓLFT is internally stable if and only if

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \text{ is invertible}$$

and

$$A_{\rm cl} := \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \text{ is Hurwitz.}$$

Internal stability of LFT, cont'd

$$A_{\text{cl}} := \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

Verify (exercise):

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} = \left[\begin{array}{c|c} A_{\text{cl}} & * \\ \hline * & * \end{array}\right] \text{ and } (I - P_{22}K)^{-1} = \left[\begin{array}{c|c} A_{\text{cl}} & * \\ \hline * & * \end{array}\right]$$

Theorem

Assume that the state-space realization for P_{22} and K are stabilizable+detectable. The ℓLFT is internally stable if and only if

- $\begin{bmatrix} I & -K(j\infty) \\ -P_{22}(j\infty) & I \end{bmatrix}^{-1}$ *is invertible, and*
- $\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1}$ *is stable*

The second item is equivalent to:

$(I - P_{22}K)^{-1}$ stable & no unstable pole/zero cancellation in forming $P_{22}K$.

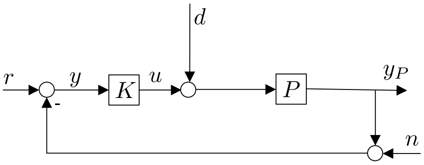
Special case

If

- the realizations of P_{22} and K are stabilizable+detectable, and
- P_{22} and K are stable,

then the ℓLFT is internally stable if and only if $\det(I - P_{22}(j\infty)K(j\infty)) \neq 0$ and $\det(I - P_{22}(s)K(s))$ has no closed RHP zeros.

Example



$$\begin{bmatrix} z \\ y \end{bmatrix} = \left[\begin{array}{ccc|c} P & -I & 0 & P \\ -P & I & -I & -P \end{array}\right] \begin{bmatrix} w \\ u \end{bmatrix}, \qquad u = Ky$$

Thus the system is internally stable if and only if:

- $I + P(j\infty)K(j\infty)$ is invertible;
- $(I+PK)^{-1}$ is stable;
- there are no hidden unstable modes in P and K, and no unstable pole-zero cancellation when forming PK.