

# Solution to Exercise 1

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1.

```
s = tf('s');
G = [1/s, 1/s];
G.InputName = 'G_in';
G.OutputName = 'G_out';
% Wr = tunableGain('Wr',1,1);
Wr = tf(1);
Wr.InputName = 'r';
Wr.OutputName = 'y1';
% Wrf = tunableGain('Wrf',1,1);
Wrf = tf(1);
Wrf.InputName = 'Wrf_in';
Wrf.OutputName = 'Wrf_out';
% Wz1 = tunableGain('Wz1',1,1);
Wz1 = tf(1);
Wz1.InputName = 'Wz1_in';
Wz1.OutputName = 'z1';
% W2 = tunableGain('W2',2,2);
W2 = tf(eye(2));
W2.InputName = 'W2_in';
W2.OutputName = 'v2';
% W1 = tunableGain('W1',2,1);
W1 = tf([1;1]);
W1.InputName = 'W1_in';
W1.OutputName = 'v1';
% Wd = tunableGain('Wd',1,1);
Wd = tf(1);
Wd.InputName = 'd';
Wd.OutputName = 'Wd_out';
% Wn = tunableGain('Wn',1,1);
Wn = tf(1);
Wn.InputName = 'n';
Wn.OutputName = 'Wn_out';
% Wz2 = tunableGain('Wz2',1,2);
Wz2 = tf(eye(2));
Wz2.InputName = 'Wz2_in';
Wz2.OutputName = 'z2';
ic1 = sumblk('y2 = Wn_out + G_out + Wd_out',1);
ic2 = sumblk('G_in = eta1 + eta2 + u',2);
ic3 = sumblk('Wz1_in = Wrf_out + G_out + Wd_out',1);
ic4 = sumblk('Wrf_in = r',1);
ic5 = sumblk('W1_in = G_out',1);
ic6 = sumblk('W2_in = u', 2);
ic7 = sumblk('Wz2_in = u',2);
input_name = {'eta1', 'eta2', 'd', 'r', 'n', 'u'};
output_name = {'v1', 'v2', 'z1', 'z2', 'y1', 'y2'};
gen_plant = connect(G, Wr, Wrf, Wz1, W2, Wd, W1, Wn, Wz2, ...
    ic1, ic2, ic3, ic4, ic5, ic6, ic7, input_name, output_name);
```

Common mistakes:

% Mistake 1

```

Wrf.InputName = 'r'; % 'r' has already been used as the InputName of Wr!
% Instead, one should use:
Wrf.InputName = 'Wrf_in';
ic4 = sumblk('Wrf_in = r');
% Mistake 2
connect(G, K, ...) % the generalized plant doesn't include K!

```

2. (1)

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

The inverse of  $G$  is proper only if  $D$  is invertible since  $G(j\infty) = D$ . In this case, we find

$$G^{-1} = \left[ \begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array} \right] \quad (1)$$

(2) Let

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

or

$$\begin{aligned}z_1 &= w_1 - Kw_2 \\ z_2 &= -Pw_1 + w_2\end{aligned}$$

To find a state-space realization for  $\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}$ , consider the state-space realization for  $P$

$$\begin{aligned}\dot{x} &= Ax + Bw_1 \\ y_1 &= Cx + Dw_1\end{aligned}$$

and  $K$

$$\begin{aligned}\dot{x}_K &= A_K x_K + B_K w_2 \\ y_2 &= C_K x_K + D_K w_2\end{aligned}$$

We have

$$\begin{aligned}z_1 &= w_1 - y_2 = w_1 - (C_K x_K + D_K w_2) \\ z_2 &= -y_1 + w_2 = -(Cx + Dw_1) + w_2\end{aligned}$$

from which we get

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix} = \left[ \begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A_K & 0 & B_K \\ \hline 0 & -C_K & I & -D_K \\ -C & 0 & -D & I \end{array} \right]$$

Now we can apply (1) to get

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \left[ \begin{array}{c|c} \frac{\begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C & 0 \end{bmatrix}}{\begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C & 0 \end{bmatrix}} & \frac{\begin{bmatrix} B & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \end{bmatrix}^{-1}}{\begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1}} \end{array} \right].$$

3. The diagram means

$$\begin{aligned}z &= P_{11}w + P_{12}u \\ y &= P_{21}w + P_{22}u \\ u &= K_{11}y + K_{12}\hat{w} \\ \hat{z} &= K_{21}y + K_{22}\hat{w}\end{aligned}$$

We need to eliminate the internal variables  $u$  and  $y$ .

$$\begin{aligned}
y &= P_{21}w + P_{22}(K_{11}y + K_{12}\hat{w}) \\
\implies y &= (I - P_{22}K_{11})^{-1}(P_{21}w + P_{22}K_{12}\hat{w}) \\
u &= K_{11}(I - P_{22}K_{11})^{-1}(P_{21}w + P_{22}K_{12}\hat{w}) + K_{12}\hat{w} \\
&= K_{11}(I - P_{22}K_{11})^{-1}P_{21}w + (K_{12} + K_{11}(I - P_{22}K_{11})^{-1}P_{22}K_{12})\hat{w}
\end{aligned}$$

Invoking the identity

$$(I - AB)^{-1}A = A(I - BA)^{-1},$$

(another useful identity

$$(I - AB)^{-1} = I + A(I - BA)^{-1}B$$

) we get

$$\begin{aligned}
K_{12} + K_{11}(I - P_{22}K_{11})^{-1}P_{22}K_{12} &= K_{12} + (I - K_{11}P_{22})^{-1}K_{11}P_{22}K_{12} \\
&= K_{12} + (I - K_{11}P_{22})^{-1}(K_{11}P_{22} - I + I)K_{12} \\
&= K_{12} - K_{12} + (I - K_{11}P_{22})^{-1}K_{12} \\
&= (I - K_{11}P_{22})^{-1}K_{12} \\
\implies u &= K_{11}(I - P_{22}K_{11})^{-1}P_{21}w + (I - K_{11}P_{22})^{-1}K_{12}\hat{w}
\end{aligned}$$

and hence

$$\begin{aligned}
z &= P_{11}w + P_{12}K_{11}(I - P_{22}K_{11})^{-1}P_{21}w + P_{12}(I - K_{11}P_{22})^{-1}K_{12}\hat{w} \\
&= (P_{11} + P_{12}K_{11}(I - P_{22}K_{11})^{-1}P_{21})w + P_{12}(I - K_{11}P_{22})^{-1}K_{12}\hat{w} \\
&= F_\ell(P, K_{11})w + P_{12}(I - K_{11}P_{22})^{-1}K_{12}\hat{w} \\
\hat{z} &= K_{21}(I - P_{22}K_{11})^{-1}(P_{21}w + P_{22}K_{12}\hat{w}) + K_{22}\hat{w} \\
&= K_{21}(I - P_{22}K_{11})^{-1}P_{21}w + (K_{22} + K_{21}(I - P_{22}K_{11})^{-1}P_{22}K_{12})\hat{w} \\
&= K_{21}(I - P_{22}K_{11})^{-1}P_{21}w + F_u(K, P_{22})\hat{w}
\end{aligned}$$

as desired.

#### 4. On the one hand

$$\begin{aligned}
\|Gu\|_2^2 &= \int_{\mathbb{R}} |y(t)|^2 dt = \int_{\mathbb{R}} |y(j\omega)|^2 d\omega \text{ (Parseval's identity)} \\
&= \int_{\mathbb{R}} |G(j\omega)u(j\omega)|^2 d\omega \\
&\leq \int_{\mathbb{R}} |G(j\omega)|_2^2 \cdot |u(j\omega)|^2 d\omega \text{ (}|G(j\omega)|_2 \text{ is the matrix 2-norm)} \\
&\leq \sup_{\omega} |G(j\omega)|_2^2 \cdot \|u\|_2^2
\end{aligned}$$

from which we deduce

$$\frac{\|Gu\|_2}{\|u\|_2} \leq \sup_{\omega \in \mathbb{R}} \{\text{largest singular value of } G(j\omega)\}, \quad \forall u \in L^2 \setminus \{0\}.$$

On the other hand, let  $\omega_0$  be such that

$$|G(j\omega_0)|_2 = \sup_{\omega} |G(j\omega)|_2$$

Let  $f \in L^2$ ,  $\|f\|_2 = 1$  be such that  $\hat{f}(j\omega)$  is concentrated around  $\omega = 0$  and  $|G(j\omega_0)|_2 = |G(j\omega_0)\hat{f}(0)|$ . Take  $u(t) = e^{-j\omega_0 t} f(t)$ . Then  $\hat{u}(j\omega) = \hat{f}(j(\omega - \omega_0))$  and

$$\begin{aligned}
\|Gu\|_2^2 &= \int_{\mathbb{R}} |G(j\omega)\hat{f}(j(\omega - \omega_0))|^2 d\omega \\
&\approx |G(j\omega_0)|_2^2.
\end{aligned}$$

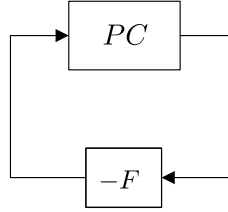
Thus  $\|Gu\|_2/\|u\|_2$  can approximate the RHS arbitrarily well. Rigorous construction is left as an exercise, see also Chapter 4.2, ZD.

5. (1) The interconnection is well-posed if and only if  $1 + PCF(j\infty)$  is invertible, or  $PCF(j\infty) \neq -1$ .  
(2) We claim that:

The system is internal stability if and only if

- a) The system is well-posed;  
b)  $(1 + PCF)^{-1}$  is stable and there is no unstable mode cancellation in forming  $PCF$ .

( $\implies$ ) Well-posedness is by definition. We prove b). If there is unstable cancellation, then this cancellation appears in  $FP$ ,  $PC$  or  $CF$ . If for example,  $FP$  has unstable mode cancellation, then  $-F$  and  $PC$  has unstable mode cancellation, which makes the system internally *unstable*, as the system can be seen as (see the box at the bottom of Page 37 of the slides):



Now the box at the bottom of Page 37 of the slides also implies that  $(1 + PCF)^{-1}$  is stable. Hence b) is proven.  
( $\impliedby$ ) If there is no unstable mode cancellation in forming  $PCF$ , then there is no hidden unstable modes in  $PC$  neither. Therefore, the system is internally stable by invoking the box at the bottom of Page 37 of the slides and the figure above.

To prove (2), we shall show that b) is equivalent to that  $N_P N_C N_F + M_P M_C M_F$  having no roots in the closed RHP.

( $\implies$ ) If  $N_P N_C N_F(s_0) + M_P M_C M_F(s_0) = 0$  for  $\text{Re}(s_0) \geq 0$ , then since

$$\frac{1}{1 + PCF} = \frac{M_C M_F M_P}{N_P N_C N_F + M_P M_C M_F}$$

is stable,  $N_P N_C N_F(s_0) = M_P M_C M_F(s_0) = 0$ . This implies that there is unstable mode cancellation in forming  $PCF$ , a contradiction.

( $\impliedby$ ) If  $N_P N_C N_F + M_P M_C M_F$  has no roots in the closed RHP, then  $(1 + PCF)^{-1}$  must be stable. Suppose there is unstable mode cancellation in forming  $PCF$ , i.e.,  $N_P N_C N_F(s_0) = M_P M_C M_F(s_0) = 0$ , for  $\text{Re}(s_0) \geq 0$ , then  $N_P N_C N_F(s_0) + M_P M_C M_F(s_0) = 0$ , i.e.,  $s_0$  is a root of  $N_P N_C N_F + M_P M_C M_F$ , a contradiction.

*Alternative proof*

**Fact:** The system is internally stable if and only if the the system is well-posed and the transfer matrix from  $[d, r, n]$  to  $[u, y, v]$  is stable.

We prove (2) by applying this fact.

( $\impliedby$ ) If  $N_P N_C N_F + M_P M_C M_F$  has no roots in the closed RHP. Then the transfer matrix is obviously stable, as each of the nice transfer functions has the form  $\frac{\text{something}}{1 + PCF}$ .

( $\implies$ ) If the system is internally stable, then by the fact above, the nice transfer functions are stable. In particular,

$$\begin{aligned} \frac{1}{1 + PCF} &= \frac{M_C M_F M_P}{N_P N_C N_F + M_P M_C M_F}, & \frac{C}{1 + PCF} &= \frac{N_C M_F M_P}{N_P N_C N_F + M_P M_C M_F} \\ \frac{PF}{1 + PCF} &= \frac{N_F N_P M_C}{N_P N_C N_F + M_P M_C M_F}, & \frac{P}{1 + PCF} &= \frac{N_P M_F M_C}{N_P N_C N_F + M_P M_C M_F} \\ & & & \dots \end{aligned}$$

are stable. If  $\exists s_0, \text{Re}(s_0) \geq 0$ , s.t.,  $N_P N_C N_F(s_0) + M_P M_C M_F(s_0) = 0$ , then

$$M_P M_C M_F(s_0) = N_P N_C N_F(s_0) = 0$$

and

$$0 = N_C M_F M_P(s_0) = N_F N_P M_C(s_0) = N_P M_F M_C(s_0) \text{ etc ...}$$

If  $s - s_0 | M_C$ , then  $s - s_0 \nmid N_C$ , and thus  $s - s_0 | M_F M_P$  since  $s - s_0 | N_C M_F M_P$ . Now, either  $s - s_0 | M_F$  or  $s - s_0 | M_P$ . If  $s - s_0 | M_F$ , then  $s - s_0 \nmid N_F$ ; by  $s - s_0 | N_C N_F M_P$  and  $s - s_0 \nmid N_C$ , we get  $s - s_0 | M_P$ . Similarly, if  $s - s_0 | M_P$ , then  $s - s_0 | M_F$ . In both cases, we get  $s - s_0 | M_C, M_F, M_P$ . But  $N_C N_F N_P$  contains  $(s - s_0)$  as a factor. Therefore, this must contradict the fact that  $(M_C, N_C), (M_F, N_F), (M_P, N_P)$  are coprime factorizations.