Bicycle Dynamics and Control

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Thanks to Richard Klein and Anders Lennartsson

Why Model?

- Insight and understanding
- Analysis, Simulation, Virtual reality
- Design optimization
- Control design
- Implementation
  - The internal model principle
  - A process model is part of the controlller
- Operator training
- Hardware in the loop simulation (e.g. Flight simulators)
- Rapid prototyping
- Diagnosis fault detection
- Multibillion dollar business
- Local: CACE, Dymola Elmqvis 1978, Dynasim (Dassault Systemes) Modelon
- Modelica - language for modeling of physical systems

Mathematical models - Uses and limitations

Solomon Wolf Golomb (1932) mathematician and engineer and a professor of electrical engineering at the University of Southern California. Best known to the general public and fans of mathematical games as the inventor of polyominoes, the inspiration for the computer game Tetris. He has specialized in problems of combinatorial analysis, number theory, coding theory and communications.

Golomb on Modeling

Distinguish at all times between the model and the real world

- Don't believe that the model is the reality
  - Don't eat the menu
  - You will never strike oil by drilling through the map
- Don't distort reality to fit the model
  - The Procrustes method
- More than one model may be useful for understanding different aspects of the same phenomenon
  - Legaized polygamy
- Don't fall in love with your model
  - Pygmalion
- Don't reject data in conflict with the model. Use such data to refute, modify or improve the model
  - Pearl Harbour

Bicycles

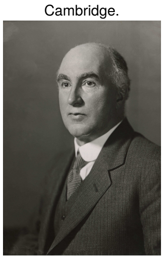
- Bicycles are convenient, environmental friendly, and efficient transportation devices
- Not trivial to explain how bicycles work. Example: Do you actively stabilize a bicycle when you ride it?
- Good example of modeling
  - From simple to complex
  - Use of models
  - Insight and understanding - how does things work
  - Design of bicycle
  - Design of wobble damper (motorcycle)
  - Autonomous bicycle
- Good illustration of many interesting issues in control
  - Modeling, stabilization, RHP poles & zeros
  - Fundamental limitations
  - Integrated process and control design
- Klein's adapted bicycles for children with disabilities

Bicycle Dynamics and Control

Bicycle Modeling

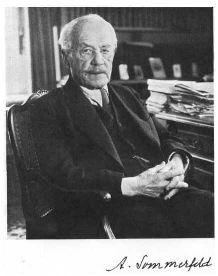
- Geometry, tires, elasticities, rider
- Early models Whipple and Carvallo 1899-1900: 4th order models
- Timoshenko-Young 1920 2nd order
- Popular thesis topics 1960-1980, manual derivations
- Rider models
- Motorcycle models Sharp 1970
- The role of software for symbolic calculation, multi-body programs and Modelica
- The control viewpoint, bicycle robots

Whipple developed his 4th order model as an undergraduate at Cambridge.



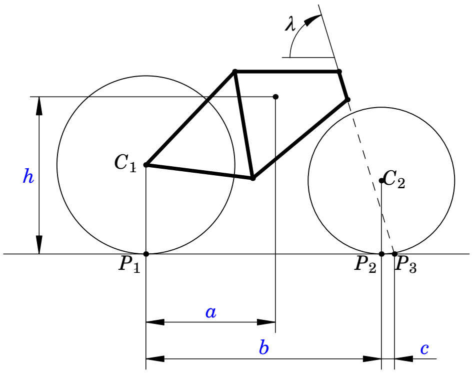
Arnold Sommerfeld on Gyroscopic Effects

That the gyroscopic effects of the wheels are very small can be seen from the construction of the wheel: if one wanted to strengthen the gyroscopic effects, one should provide the wheels with heavy rims and tires instead of making them as light as possible. It can nevertheless be shown that these weak effects contribute their share to the stability of the system.



Four of Sommerfeld's graduate students got the Nobel Prize Heisenberg 1932, Debye 1936, Pauli 1945 and Bethe 1962

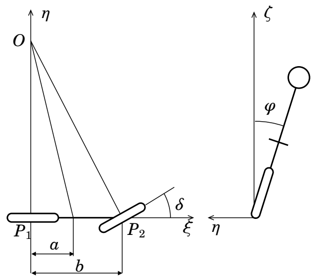
Geometry



Tilt Dynamics

Assume all angles small. Angular momentum and torque along ζ axis

$$\mathcal{M}_\xi = J_{\xi\xi}\omega_\xi - D_{\xi\zeta}\omega_\zeta$$
$$= J\frac{d\varphi}{dt} - D\frac{V_0}{b}\delta$$
$$D = mah$$
$$T_\xi = mgh\varphi + m\frac{V_0^2}{b}\delta$$



$$\frac{d\mathcal{M}_\xi}{dt} = T_\xi \quad \Rightarrow \quad J\frac{d^2\varphi}{dt^2} - \frac{mahV_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mhV_0^2}{b}\delta$$

Compare with inverted pendulum!

The Inverted Pendulum Model $\delta \rightarrow \varphi$

Linearized tilt dynamics

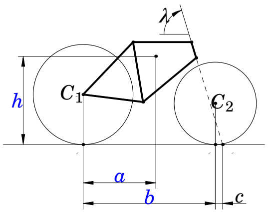
$$J\frac{d^2\varphi}{dt^2} - \frac{mahV_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mhV_0^2}{b}\delta$$

Model that relates steering angle δ to tilt φ

$$\frac{d^2\varphi}{dt^2} - \frac{mgh}{J}\varphi = \frac{mhV_0^2}{bJ}\delta + \frac{amhV_0}{bJ}\frac{d\delta}{dt}$$

Transfer function:

$$P(s) = \frac{amhV_0}{bJ}\frac{s + V_0/a}{s^2 - mgh/J}$$
$$= \frac{hV_0}{br^2}\frac{as + V_0}{s^2 - \sqrt{gh/r^2}}$$



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2. Modeling
3. Stabilization
4. Rear wheel steering
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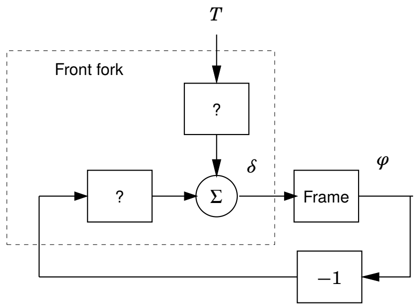
Some Interesting Questions

- How do you stabilize a bicycle?
  By steering or by leaning?
- Do you normally stabilize a bicycle when you ride it?
- Why is it possible to ride no hands
- How is stabilization influenced by the design of the bike?
- Why does the front fork look the way it does?
- The main message:
  A bicycle is a feedback system!
  The front fork is the key!
- Is the control variable steering angle or steering torque?

Block Diagram of a Bicycle

Control variable: Handlebar torque T

Process variables: Steering angle δ, tilt angle φ

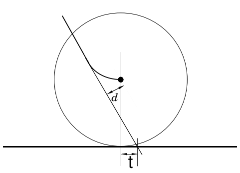


A feedback system

The Front Fork

The front fork has many interesting features that were developed over a long time. Its behavior is complicated by geometry, the trail, tire-road interaction and gyroscopic effects. We will describe it by a strongly simplified static linear model.

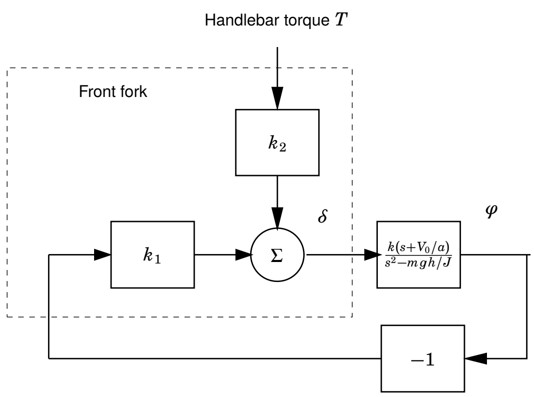
With a positive trail the front wheel lines up with the velocity (caster effect). The trail also creates a torque that turns the front fork into the lean. A static torque balance gives



$$T - mgt\varphi - mgt\alpha\delta = 0$$
$$\delta = -k_1\varphi + k_2 T$$

Qualitative experimental verification. In reality more complex, dynamics and velocity dependence will be discussed later.

Block Diagram of a Bicycle



The Closed Loop System

Combining the equations for the frame and the front fork gives

$$\frac{d^2\varphi}{dt^2} = \frac{mgh}{J}\varphi + \frac{amhV_0}{bJ}\frac{d\delta}{dt} + \frac{mhV_0^2}{bJ}\delta$$
$$\delta = -k_1\varphi + k_2 T$$

we find that the closed loop system is described by

$$\frac{d^2\varphi}{dt^2} + \frac{amhk_1V_0}{bJ}\frac{d\varphi}{dt} + \frac{mgh}{J}\Big(\frac{k_1V_0^2}{bg} - 1\Big)\varphi = \frac{amk_2hV_0}{bJ}\Big(\frac{dT}{dt} + \frac{V_0}{a}T\Big)$$

This equation is stable if

$$V_0 > V_c = \sqrt{bg/k_1}$$

where V_c is the critical velocity. Physical interpretation. Think about this next time you bike!

Stabilization

The bicycle is a feedback system. The clever design of the front fork gives a feedback because a the front wheel will steer into a lean. The closed loop system can be described by the equation

$$\frac{d^2\varphi}{dt^2} + \frac{amhk_1V_0}{bJ}\frac{d\varphi}{dt} + \frac{mgh}{J}\Big(\frac{k_1V_0^2}{bg} - 1\Big)\varphi = \frac{amk_2hV_0}{bJ}\Big(\frac{dT}{dt} + \frac{V_0}{a}T\Big)$$

which shows how tilt angle φ depends on handle bar torque T.

The equation is unstable for low speed but stable for high speed $V_0 > V_c = \sqrt{bg/k_1}$, the critical velocity.

This means that the bicycle is self-stabilizing if the velocity is larger than the critical velocity V_c! You can observe this by rolling a bicycle down a gentle slope or by biking at different speeds.

Bicycle Dynamics and Control

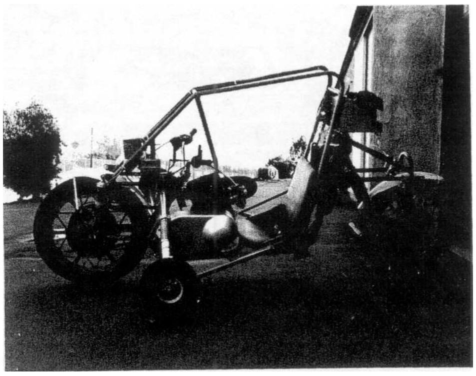
Rear Wheel Steering

F. R. Whitt and D. G. Wilson (1974) Bicycling Science - Ergonomics and Mechanics. MIT Press Cambridge, MA.

Many people have seen theoretical advantages in the fact that front-drive, rear-steered recumbent bicycles would have simpler transmissions than rear-driven recumbents and could have the center of mass nearer the front wheel than the rear. The U.S. Department of Transportation commissioned the construction of a safe motorcycle with this configuration. It turned out to be safe in an unexpected way: No one could ride it.

The Santa Barbara Connection

The NHSA Rear Steered Motorcycle



Comment by Robert Schwarz

The outriggers were essential; in fact, the only way to keep the machine upright for any measurable period of time was to start out down on one outrigger, apply a steer input to generate enough yaw velocity to pick up the outrigger and then attempt to catch it as the machine approached vertical. Analysis of film data indicated that the longest stretch on two wheels was about 2.5 s.

The Linearized Tilt Equation

Front wheel steering:

$$\frac{d^2\varphi}{dt^2} = \frac{mgh}{J}\varphi + \frac{amhV_0}{bJ}\frac{d\delta}{dt} + \frac{mhV_0^2}{bJ}\delta$$

Rear wheel steering (change sign of V_o):

$$\frac{d^2\varphi}{dt^2} = \frac{mgh}{J}\varphi - \frac{amhV_0}{bJ}\frac{d\delta}{dt} + \frac{mhV_0^2}{bJ}\delta$$

The transfer function of the system is

$$P(s) = \frac{amhV_0}{bJ}\frac{-s + \frac{V_0}{a}}{s^2 - \frac{mgh}{J}}$$

One pole and one zero in the right half plane.

The Transfer Function

$$P(s) = \frac{amhV_0}{bJ}\frac{-s + \frac{V_0}{a}}{s^2 - \frac{mgh}{J}}$$

One RHP pole at $p = \sqrt{\frac{mgh}{J}} \approx 3$ rad/s (the pendulum pole)

One RHP zero at $z = \frac{V_0}{a} \approx 5, \quad \frac{z}{p} = \frac{5}{3} \approx 1.7, \quad M_s \geq 4$

Pole position independent of velocity but zero proportional to velocity. When velocity increases from zero to high velocity you pass a region where $z = p$ and the system is unreachable.

Does Feedback from Rear Fork Help?

Combining the equations for the frame and the rear fork gives

$$\frac{d^2\varphi}{dt^2} = \frac{mgh}{J}\varphi - \frac{amhV_0}{bJ}\frac{d\delta}{dt} + \frac{mhV_0^2}{bJ}\delta$$
$$\delta = -k_1\varphi + k_2 T$$

we find that the closed loop system is described by

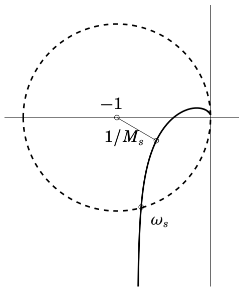
$$\frac{d^2\varphi}{dt^2} - \frac{amhk_1V_0}{bJ}\frac{d\varphi}{dt} + \frac{mgh}{J}\Big(\frac{k_1V_0^2}{bg} - 1\Big)\varphi = \frac{amk_2hV_0}{bJ}\Big(\frac{dT}{dt} + \frac{V_0}{a}T\Big)$$

where $V_c = \sqrt{bg/k_1}$. This equation is unstable for all k_1. There are several ways to turn the rear fork but it makes little difference.

Can the system be stabilized robustly with a more complex controller?

Can a general linear controller help?

Nyquist's stability theorem



The sensitivity function

$$S = \frac{1}{1+L}$$

For a system with a pole p and a zero z in the right half plane the maximum modulus theorem implies

$$M_s = \max_{\omega} |S(i\omega)| \geq \frac{|z+p|}{|z-p|}$$

$|S(i\omega)| < 2$ implies $z > 3p$ (or $z < p/3$) for any controller!

Return to Rear Wheel Steering ...

The zero-pole ratio is

$$\frac{z}{p} = \frac{V_0\sqrt{J}}{a\sqrt{mgh}} = \frac{V_0\sqrt{J_{cm} + mh^2}}{a\sqrt{mgh}}$$

The system is not controlable if $z = p$, and it cannot be controlled robustly if the ratio z/p is in the range of 0.3 to 3.

To make the ratio large you can

- Make a small by leaning forward $v_0 \geq a\sqrt{\frac{g}{h}}\frac{M_s+1}{M_s-1}$
- Make V_0 large by biking fast (takes guts)
- Make J large by standing upright
- Sit down, lean back when the speed is sufficiently large

Klein's Unridable Bike

Klein's Ridable Bike

The Lund University Unridable Bike

The UCSB Rideable Bike

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Steering and Stabilization - A Classic Problem

Lecture by Wilbur Wright 1901:

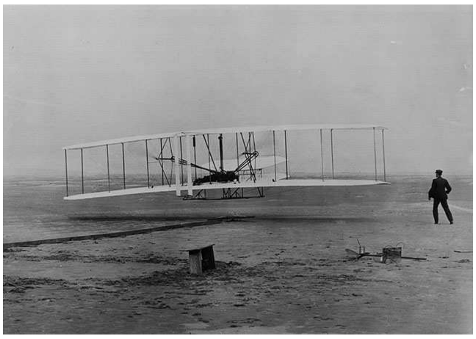
Men know how to construct air-planes.
Men also know how to build engines.
Inability to balance and steer still confronts
students of the flying problem.
When this one feature has been worked out,
the age of flying will have arrived, for
all other difficulties are of minor importance.

The Wright Brothers figured it out and flew the Wright Flyer at Kitty Hawk on December 17 1903!

Ship steering: Minorsky 1922: *It is an old adage that a stable ship is difficult to steer.*

Birds: John Maynard Smith 1955: *To a flying animal there are great advantages to be gained by instability. Among the most obvious is manoeuvrability.*

The Wright Flyer - Unstable but Maneuvrable

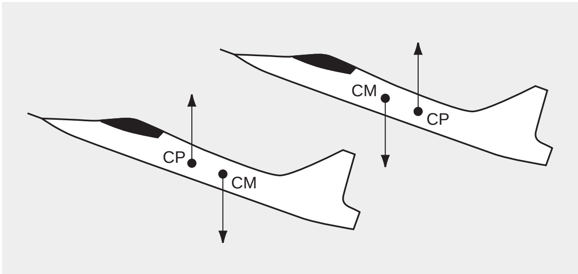


Draper on Wright

The Wright Brothers rejected the principle that aircraft should be made inherently so stable that the human pilot would only have to steer the vehicle, playing no part in stabilization. Instead they deliberately made their airplane with negative stability and depended on the human pilot to operate the movable surface controls so that the flying system - pilot and machine - would be stable. This resulted in and increase in manoeuvrability and controllability.

The 43rd Wilbur Wright Memorial Lecture before the Royal Aeronautical Society, May 19 1955.

JAS Gripen



Birds

The earliest birds pterosaurs, and flying insects were stable. This is believed to be because in the absence of a highly evolved sensory and nervous system they would have been unable to fly if they were not. To a flying animal there are great advantages to be gained by instability. Among the most obvious is manoeuvrability. It is of equal importance to an animal which catches its food in the air and to the animals upon which it preys. It appears that in the birds and at least in some insects the evolution of the sensory and nervous systems rendered the stability found in earlier forms no longer necessary.

John Maynard Smith The Importance of the nervous sytem in the evolution of aimal flight. Evolution, 6 ,(1952) 127-9.

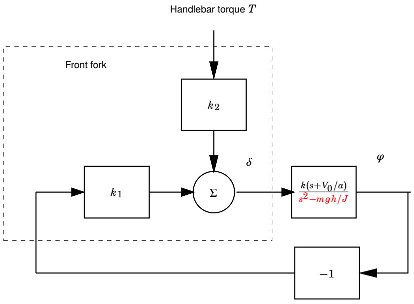
Steering

Having understood stabilization of bicycles we will now investigate steering for the bicycle with a rigid rider.

- Key question: How is the path of the bicycle influenced by the handle bar torque?
- Steps in analysis, find the relations
  - How handle bar torque influences steering angle
  - How steering angle influences velocity
  - How velocity influences the path

We will find that the instability of the bicycle frame causes some difficulties in steering (dynamics with right half plane zeros). This has caused severe accidents for motor bikes.

How Steer Torque Influences Steer Angle



Transfer function from T to δ is

$$\frac{k_2}{1+k_1P(s)} = \frac{k_2}{1+k_1\frac{k(s+V_0/a)}{s^2-mgh/J}} = k_2\frac{s^2-mgh/J}{s^2+\frac{amhk_1V_0}{bJ}s+\frac{mgh}{J}\left(\frac{V_0^2}{V_c^2}-1\right)}$$

Summary of Equations

Kinematics

$$\frac{dy}{dt} = V\psi$$
$$\frac{d\psi}{dt} = \frac{V}{b}\delta.$$

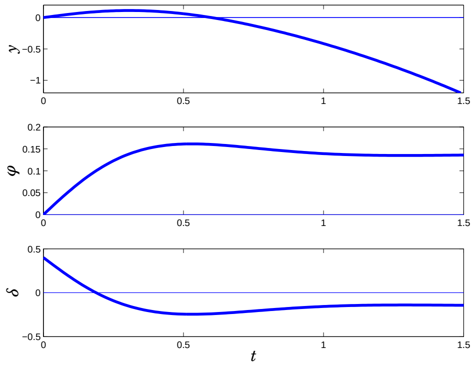
The transfer function from steer angle δ to path deviation y is

$$G_{y\delta}(s) = \frac{V^2}{bs^2}.$$

Transfer function from steer torque T to y

$$G_{yT}(s) = \frac{k_1V^2}{b}\frac{s^2-mgh/J}{s^2\left(s^2+\frac{k_2VD}{bJ}s+\frac{mgh}{J}\left(\frac{V^2}{V_c^2}-1\right)\right)}.$$

Simulation



Summary

- The simple inverted pendulum model with a rigid rider can explain stabilization. The model indicates that steering is difficult due to the right half plane zero in the transfer function from handle bar torque to steering angle.
- The right half plane zero has some unexpected consequences which gives the bicycle a counterintuitive behavior. This has caused many motorcycle accidents.
- How can we reconcile the difficulties with our practical experience that a bicycle is easy to steer?
- The phenomena depends on the assumption that the rider does not lean.
- The difficulties can be avoided by introducing an extra control variable (leaning).

Wilbur Wright on Counter-Steering

> *I have asked dozens of bicycle riders how they turn to the left. I have never found a single person who stated all the facts correctly when first asked. They almost invariably said that to turn to the left, they turned the handlebar to the left and as a result made a turn to the left. But on further questioning them, some would agree that they first turned the handlebar a little to the right, and then as the machine inclined to the left they turned the handlebar to the left and as a result made the circle inclining inwardly.*

Wilbur's understanding of dynamics contributed significantly to the Wright brothers' success in making the first airplane flight.

Adding an input (lean) eliminates the RHP zero!

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Overview of Models

- Second order linear model - inverted pendulum + static front fork
- Fourth order linear model (Carvallo 1897-1900 Whipple 1889)
- Fourth order nonlinear model
- Wheels and nonholonomic systems
- Flexible tires
- Tire road interaction
- Frame flexibility
- Rider model
- Multi-body software useful
- There is a Modelica library for bicycles

Carvallo-Whipple 4th Order Linear Model

This model can be derived in different ways, Newton's equations, Lagrange's equations, projection methods etc. Calculations are complicated and error prone. Versions of the model are found in

- Whipple 1899
- Carvallo 1897-1900
- Klein and Sommerfeld 1910
- Neimark Fufaev 1968
- Many doctoral theses 1970-1990
- Schwab et al 2004

Parameters for 4th Order Linear Model

The model is described by 25 parameters; wheel base $b = 1.00$ m, trail $c = 0.08$, head angle $\lambda = 70°$, wheel radii $R_{rw} = R_{fw} = 0.35$, and data in the table.

	Rear Frame	Fr Frame	Rr Wheel	Fr Wheel
Mass m [kg]	87 (12)	2	1.5	1.5
Center of Mass				
x [m]	0.492 (0.439)	0.866	0	b
z [m]	1.028 (0.579)	0.676	R_{rw}	R_{fw}
Inertia Tensor				
J_{xx} [kg m^2]	3.28 (0.476)	0.08	0.07	0.07
J_{xz} [kg m^2]	-0.603 (-0.274)	0.02	0	0
J_{yy} [kg m^2]	3.880 (1.033)	0.07	0.14	0.14
J_{zz} [kg m^2]	0.566 (0.527)	0.02	J_{xx}	J_{xx}

A Fourth Order Linear Model

Momentum balances for frame and front fork

$$M\begin{pmatrix}\ddot\varphi\\ \ddot\delta\end{pmatrix} + CV\begin{pmatrix}\dot\varphi\\ \dot\delta\end{pmatrix} + (K_0 + K_2V^2)\begin{pmatrix}\varphi\\ \delta\end{pmatrix} = \begin{pmatrix}0\\ T\end{pmatrix},$$

Notice structure of velocity dependence. The matrices are

$$M = \begin{pmatrix}96.8\ (6.00) & -3.57(-0.472)\\ -3.57\ (-0.472) & 0.258\ (0.152)\end{pmatrix},$$

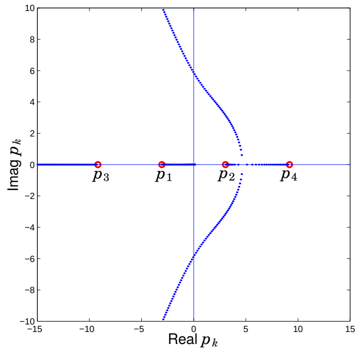
$$C = \begin{pmatrix}0 & -50.8\ (-5.84)\\ 0.436\ (0.436) & 2.20\ (0.666)\end{pmatrix},$$

$$K_0 = \begin{pmatrix}-901.0\ (-91.72) & 35.17\ (7.51)\\ 35.17\ (7.51) & -12.03\ (-2.57)\end{pmatrix},$$

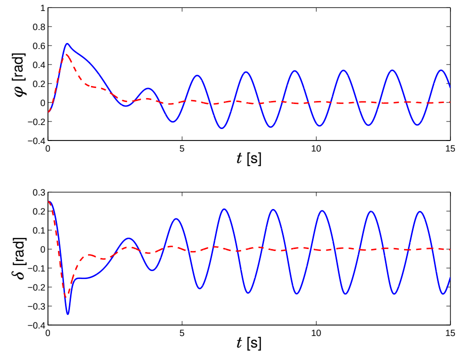
$$K_2 = \begin{pmatrix}0 & -87.06\ (-9.54)\\ 0 & 3.50\ (0.848)\end{pmatrix}.$$

Data without rider in parantheses

Root Locus Bicycle with Rider



Simulation of Fourth Order Nonlinear Model



Open Problems

The nonlinear model has very rich behavior which has not been fully explored.

- Local behavior around all equiblibira
- Other equilibria of interest are steady turning

Movies of Weave and Wobble

IsleOfManMC

wobble.mpeg

weave.mpg

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Robot Bicycles

- 1988 Klein UIUC
- 1996 Pacejka Delft motorcycle robot
- 2004 Tanaka and Murakami
- 2005 UCSB
- 2005 Yamakita and Utano Titech
- 2005 Murata Co

Murata Manufacturing Company
Japan Times Oct 5 2005

Klein's Adapted Bikes for Children with Disabilities

Over a dozen clinics for children and adults with a wide range of disabilities, including Down syndrome, autism, mild cerebral palsy and Asperger's syndrome. More than 2000 children aged 6-20 have been treated, see http://www.losethetrainingwheels.org

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Conclusions

- Bicycle dynamics is a good illustration of modeling theoretically and experimentally
  - Much insight into stabilization and steering can be derived from simple models
  - Interaction of system and control design (the front fork)
  - Counterintuitive behavior because of dynamics with right half plane zeros
  - Importance of several control variables
- Lesson 1: Dynamics is important! Things may look OK statically but intractable because of dynamics.
- Lesson 2: A system that is difficult to control because of zeros in the right half plane can be improved significantly by introducing more control variables (steer and lean).

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