Mechanical Systems

1. Introduction
2. Astronomy
3. Newton, Lagrange, Hamilton and Jacobi
4. Pendulum on a Cart
5. Furuta Pendulum
6. Ball and Beam
7. Summary

Natural Science and Engineering Science

Many similarities but also many differences

Natural Phenomena	**Technical Systems**
Insight	Insight
Understanding	Understanding
Analysis	Synthesis
Isolation	Interaction
Fundamental Laws	System Principles

Feedback is a a good system principle!
Recall the Magic of Feedback

The Magic of Feedback

Good properties:

- Attenuate effects of disturbances - process control, automotive
- Make good systems from bad components - feedback amplifier
- Follow command signals - robotics, automotive
- Stabilize and shape behavior - flight control

Bad properties:

- Feedback may cause instability
- Feedback feeds measurement noise into the system

Arthur C. Clarke: Any sufficiently advanced technology is indistinguishable from magic

Modeling - a very Rich Field

- Mechanical systems
  - Classical mechanics
  - Microsystems
  - Mechatronics
- Fluid systems
- Thermal systems
- Thermofluid
- Electric circuits
  - Resistors
  - Capacitors
  - Transformers
  - Networks
- Electronics
- Electromechanical systems
  - Motors
  - Generators
- Vehicles
  - Bicycles and cars
  - Ships
  - Airplanes and rockets
- Power systems
  - Steam generators
  - Hydro-electric
  - Networks
- Chemical processes
  - Reactors
  - Distillation columns
- Biological systems
  - Compartment models
  - Pharmacokinetics
- Ecosystems
- Economics

Mechanical Systems

- Mechanical systems a cornerstone of all engineering education
- Many examples
  - Lab systems (pendula)
  - Robotics stationary and mobile
  - Important elements of vehicles
- Mechatronics
- Micromechanics
- Sensors
  - Gyros and accelerometers

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Astronomy - Role Model for Natural Science

Astronomy is on of the oldest natural sciences. It investigates celestial objects such as planets, moons and stars. The early civilizations in recorded history made methodical observations of the night sky. These include the Babylonians, Greeks, Indians, Egyptians, Chinese, Maya, and many ancient indigenous peoples of the Americas.

- Experiments
- Ulugh Beg grandson of Timur Lenk Observatory in Samarkand 1420
- Tycho Brahe 1546-1601
- Hans Lippershey 1608 patent for a refracting telescope.
- Galileo 1609
- Kepler 1571-1630
- Theory emerges
- Newton 1643-1727

Key Problem

Predict the future positions of the planets.

- The emergence of Natural Science
- Conceptual insight
- How do they move
- What causes the motion
- How can be motion be described
- Abstractions Physical Laws
- Interplay of observation and theory
- Recently extremely precise measurements
- Recently verifiction of Einsteins relativity throry

A prime example of a modeling effort that spanned many centuries with brillant contributors and revolutionary consequences.

An Example - The Giant Modelers

The different phases

- Early Observations: Tycho Brahe and Timur Lenk
- Finding features: Kepler
- Theory development: Newton
- Improved data treatment: Gauss
- Abstraction: Euler, Lagrange, Hamilton and Jacobi
- Further abstractions: Poincare, Birkhoff
- Recent contributions Smale, Arnold and Chaos

Astronomical Observations

- Optical astronomy
  - Earth bound
  - In space Hubble
- Radio astronomy
  - Five-hundred-meter Aperture Spherical radio Telescope (FAST) Chinese:, nicknamed Tianyan (lit. "Sky's/Heaven's Eye"). FAST has a 500 m diameter dish constructed in a natural depression in the landscape.
- Combining antennas over the world
- On 11 February 2016 it was announced that the LIGO collaboration had directly observed gravitational waves for the first time in September 2015. The second observation of gravitational waves was made on 26 December 2015 and announced on 15 June 2016. Barry Barish, Kip Thorne and Rainer Weiss were awarded the 2017 Nobel Prize in Physics for leading this work.

Features from Observation

Tycho Brahe was mathematician at the court of Emperor Rudolf II in Prag, Kepler was his assistant. Brahe reluctantly gave Kepler data for Mars, the planet whose path deviates most from a circle.
By analysis of the data Kepler found three laws.

1. Planets move in ellipses with the sun at the center
2. Equal areas are covered in equal times
3. Time to go around the sun related to the size of the orbit
4. Keplers formula

$$M = E - e \sin E$$

The Three Body Problem - Poincare

Newton could solve his equations for two bodies, the sun and the earth, and obtain ellipsoidal orbits.
Efforts to solve the equations for three planets failed. Poincare gave a new view. He emphasized the qualitative aspects and started a new vigorous development.

The Planet Ceres and Karl Friedrich Gauss

The dwarf planet Ceres is the largest object in the asteroid belt between Mars and Jupiter, and it's the only dwarf planet located in the inner solar system. It was the first member of the asteroid belt to be discovered when Giuseppe Piazzi spotted it in 1801. The planet was lost after 41 days of observations because it had an almost circular orbit. Gauss decided to find in and invented the least squares method. Ceres was rediscovered in January 1802. Gauss was polishing his manuscript and published in 1809.

Carl Friedrich Gauss
1777-1855

Karl Friedrich Gauss and Least Squares

The story of the planet Ceres, discovered in 1781, almost circular orbit. Vanished from view. Recovered by Gauss method in 1802. K. F. Gauss Teoria Motus Corporum Coelestium 1809.
"The most probable values of the unknown parameters, are those which minimize the sum of the squares of the differences between the observed and computed values."
"The principle that the sum of the squares of the differences between observed and computed quantities must be a minimum may be considered independently of the calculus of probabilities."
"Instead of using the sum of squares (our principle) we could use sum of any even power of the errors. But of all these principles ours is the most simple."

Laser Interferometer Gravitational-Wave Observatory (LIGO)

The LIGO concept built upon early work by many scientists to test a component of Albert Einstein's theory of general relativity, the existence of gravitational waves.
In 1967 Rainer Weiss of MIT published an analysis of interferometer use and initiated the construction of a prototype which was never completed.
The current LIGO multi-kilometer-scale gravitational wave detectors uses laser interferometry to measure the minute ripples in space-time caused by passing gravitational waves from cataclysmic cosmic events such as colliding neutron stars or black holes, or by supernovae. It consists of two widely-separated interferometers within the United States—one in Hanford, Washington and the other in Livingston, Louisiana—operated in unison to detect gravitational waves.
The Nobel Prize in Physics 2017 was divided, one half awarded to Rainer Weiss, the other half jointly to Barry C. Barish and Kip S. Thorne "for decisive contributions to the LIGO detector and the observation of gravitational waves."

Mechanical Systems

Newton a Modeling Giant

Newton investigated the motion of two planets subject to a gravitational force. He formulated the law for gravitation

$$F = k\frac{mM}{r^2}$$

and he also formulated the law of momentum balance

$$\frac{d}{dt}mv = F, \qquad m\frac{d^2x}{dt^2} = F$$

and the analog for angular momentum.
He also developed differential calculus to be able to manipulate the equations.
The theory that emerged covered much more than the original problem.
The three-body problem defied analysis.

Lagrange's Equation

Introduce

q generalized coordinates

p generalized momenta

Compute

Potential energy $V(q)$

Kinetic energy $T(p,q)$

Lagrangian $L = T - V$

Equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$$

Joseph-Louis Lagrange
1813-1976

Hamilton's Equations

Let q be the generalized coordinates, the hamiltionian is the total energy of the system

$$H(p,q) = V(q) + T(q)$$

Equations of motion are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

The system moves so that the total energy is minimal!

William Rowan Hamilton
1805-1865 Algebra of quaternions

The Hamilton-Jacobi Equation

Let q be the generalized coordinates and let $H(q,p) = T(p,q) + V(q)$ be the Hamiltionian (total energy). The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + H\big(q, \frac{\partial S}{\partial q}\big) = 0$$

Compare Pontryagins Maximum Principle!

$$H(x,p,u) = V(x,u) + p^T f(x,u)$$
$$H_0(x,p) = min_u H(x,p,u)$$
$$\frac{\partial S}{\partial t} + H_0\Big(x, \frac{\partial S}{\partial x}\Big) = 0$$
$$\frac{dq}{dt} = \frac{\partial H_0}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H_0}{\partial q}$$

The Hamilton-Jacobi-Bellman Equation

$$\frac{dx}{dt} = f(x,u), \quad \min_u V(x.u)$$

The Hamiltonian

$$H(x,p,u) = V(x,u) + p^T f(x,u)$$
$$H_0(x,p) = min_u H(x,p,u)$$

Hamilton-Jacobi-Bellman equation

$$\frac{\partial S}{\partial t} + H_0\big(x, \frac{\partial S}{\partial x}\big) = 0$$

Pontryagins maximum principle

$$\frac{dx}{dt} = \frac{\partial H_0}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H_0}{\partial x}$$

Richard Ernest Bellman
1920-1984

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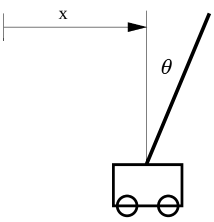
Pendulum on a Cart

$$x_p = x + l\sin\theta$$
$$y_p = l\cos\theta$$
$$\dot{x}_p = \dot{x} + l\dot{\theta}\cos\theta$$
$$\dot{y}_p = -l\dot{\theta}\sin\theta$$



Potential energy

$$V = mgl\cos\theta$$

Kinetic energy

$$\begin{aligned} T &= \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m_{cart}\dot{x}^2 + \frac{1}{2}m(\dot{x}_p^2 + \dot{y}_p^2) \\ &= \frac{1}{2}(J + ml^2)\dot{\theta}^2 + \frac{1}{2}(m_{cart} + m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta \\ &= \frac{1}{2}J_p\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta \end{aligned}$$

Equations of Motion

$$L = T - V = \frac{1}{2}J_p\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta - mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = J_p\dot{\theta} + ml\dot{x}\cos\theta \qquad \frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta$$
$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + ml\dot{\theta}\cos\theta \qquad \frac{\partial L}{\partial x} = 0$$

Lagranges Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$$

give the following equations of motion

$$J_p\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0$$
$$ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + M\ddot{x} = F$$

Assessment

Does the equation

$$J_p\ddot\theta + ml\ddot x\cos\theta - mgl\sin\theta = 0$$
$$ml\ddot\theta\cos\theta - ml\dot\theta^2\sin\theta + M\ddot x = F$$

make sense?

- Interpretation of the different terms
- What happens if the cart is very heavy? (Hint $M = m + m_{cart}$)
- Can we find a suitable normalization?
- When can the interaction between pendulum and cart be neglected?
- How many independent parameters are there?

Normalisation 1

$$J_p\ddot\theta + ml\ddot x\cos\theta - mgl\sin\theta = 0$$
$$ml\ddot\theta\cos\theta - ml\dot\theta^2\sin\theta + M\ddot x = F$$

Divide first equation with J_p and the second with M. Hence

$$\ddot\theta + \frac{ml}{J_p}\ddot x\cos\theta - \frac{mgl}{J_p}\sin\theta = 0$$
$$\frac{ml}{M}\ddot\theta\cos\theta - \frac{ml}{M}\dot\theta^2\sin\theta + \ddot x = \frac{F}{M}$$

Four parameters $\frac{ml}{J_p}$, $\frac{ml}{M}$, $\omega_0 = \sqrt{mgl/J_p}$ and $1/M$! Notice that all parameters are not dimension free!

Normalisation 2

Use

- $1/\omega_0$ as a time scale
- l as a length scale
- $1/(Ml)$ as unit of force (u is acceleration)
- $\alpha = \dfrac{ml^2}{J_p} = \dfrac{ml^2}{ml^2 + J}$
- $\beta = \dfrac{m}{M}$

Then

$$\ddot\theta + \alpha\ddot x\cos\theta - \sin\theta = 0$$
$$\beta\ddot\theta\cos\theta - \beta\dot\theta^2\sin\theta + \ddot x = u$$

Only two parameters!!

Linearisation

Normalized and scaled equations of motion

$$\ddot\theta + \alpha\ddot x\cos\theta - \sin\theta = 0$$
$$\beta\ddot\theta\cos\theta - \beta\dot\theta^2\sin\theta + \ddot x = u$$

Linearize

$$\ddot\theta + \alpha\ddot x\cos\theta_0 - \theta\cos\theta_0 = 0$$
$$\beta\ddot\theta\cos\theta_0 + \ddot x = u$$

Notice sign changes for the equilibria

$$\alpha = \frac{ml^2}{J_p} = \frac{ml^2}{ml^2 + J}$$
$$\beta = \frac{m}{M}$$

Stabilizing a Pendulum on Cart

Start with normalized equations in linearized form

$$\ddot\theta + \alpha\ddot x\cos\theta_0 - \theta\cos\theta_0 = 0$$
$$\beta\ddot\theta\cos\theta_0 + \ddot x = u$$

Eliminate x!

$$(1 - \alpha\beta\cos^2\theta_0)\ddot\theta - \theta\cos\theta_0 = -\alpha u\cos\theta_0$$

where

$$\alpha\beta = \frac{ml^2}{J + ml^2}\frac{m}{M}$$

Notice that $1 - \alpha\beta$ does not change sign.

Stabilizing the Pendulum

$$(1 - \alpha\beta\cos^2\theta_0)\ddot\theta - \theta\cos\theta_0 = -\alpha u\cos\theta_0$$

Up position ($\theta_0 = 0$)

$$(1 - \alpha\beta)\ddot\theta - \theta = -\alpha u$$

Down position ($\theta_0 = \pi$)

$$(1 - \alpha\beta)\ddot\theta + \theta = \alpha u$$

A PD controller will do the job. It adds a terms θ and $\dot\theta$.
Safe to experiment in down position!

PD Control

$$\begin{aligned} \text{up} \quad & (1-\alpha\beta)\ddot\theta - \theta = -\alpha u \\ \text{down} \quad & (1-\alpha\beta)\ddot\theta + \theta = \alpha u \\ \text{PD} \quad & \alpha u = k\theta + k_v\dot\theta \end{aligned}$$

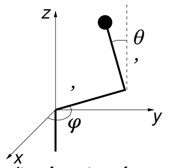
Closed loop

$$\begin{aligned} \text{up} \quad & (1-\alpha\beta)\ddot\theta + k_v\dot\theta + (k-1)\theta = 0 \\ \text{down} \quad & (1-\alpha\beta)\ddot\theta - k_v\dot\theta + (1-k)\theta = 0 \end{aligned}$$

Mechanical Systems

The Furuta Pendulum

$$
\begin{aligned}
x_{cm} &= r\cos\varphi - \ell\sin\theta\sin\varphi\\
y_{cm} &= r\sin\varphi + \ell\sin\theta\cos\varphi\\
z_{cm} &= \ell\cos\theta\\
\dot x_{cm} &= -r\dot\varphi\sin\varphi - \ell\dot\varphi\cos\varphi\sin\theta - \ell\dot\theta\sin\varphi\cos\theta\\
\dot y_{cm} &= r\dot\varphi\cos\varphi - \ell\dot\varphi\sin\varphi\sin\theta + \ell\dot\theta\cos\varphi\cos\theta\\
\dot y_{cm} &= -\ell\dot\theta\sin\theta
\end{aligned}
$$



Velocity of center of mass of swinging pendulum

$$v^2 = r^2\dot\varphi^2 + \ell^2\dot\varphi^2\sin^2\theta + 2r\ell\dot\varphi\dot\theta\cos\theta + \ell^2\dot\theta^2$$

Equations of Motion

Kinetic Energy

$$
\begin{aligned}
2T &= mv^2 + J_b\dot\varphi^2 + J(\dot\theta^2 + \dot\varphi^2\sin^2\theta)\\
&= (J_b + mr^2 + (J + m\ell^2)\sin^2\theta)\dot\varphi^2 + mr\ell\dot\varphi\dot\theta\cos\theta + (J + m\ell^2)\dot\theta^2\\
&= (J_a + J_p\sin^2\theta)\dot\varphi^2 + mr\ell\dot\varphi\dot\theta\cos\theta + J_p\dot\theta^2
\end{aligned}
$$

where

$$
\begin{aligned}
J_p &= J + m\ell^2\\
J_a &= J_b + mr^2
\end{aligned}
$$

Potential Energy

$$V = mg\ell\cos\theta$$

Equations of Motion ...

$$
\begin{aligned}
\frac{\partial L}{\partial\theta} &= -mr\ell\dot\varphi\dot\theta\sin\theta + J_p\dot\varphi^2\sin\theta\cos\theta + mg\ell\sin\theta\\
\frac{\partial L}{\partial\dot\theta} &= J_p\dot\theta + mr\ell\dot\varphi\cos\theta\\
\frac{\partial L}{\partial\varphi} &= 0\\
\frac{\partial L}{\partial\dot\varphi} &= mr\ell\dot\theta\cos\theta + (J_a + J_p\sin^2\theta)\dot\varphi
\end{aligned}
$$

Equations of motion

$$
\begin{aligned}
&J_p(\ddot\theta - \dot\varphi^2\sin\theta\cos\theta) + mr\ell\ddot\varphi\cos\theta - mg\ell\sin\theta = 0\\
&mr\ell\ddot\theta\cos\theta - mr\ell\dot\theta^2\sin\theta + 2J_p\dot\theta\dot\varphi\sin\theta\cos\theta + (J_a + J_p\sin^2\theta)\ddot\varphi = u.
\end{aligned}
$$

Interpretation of the Equations

$$
\begin{aligned}
&J_p(\ddot\theta - \dot\varphi^2\sin\theta\cos\theta) + mr\ell\ddot\varphi\cos\theta - mg\ell\sin\theta = 0\\
&mr\ell\ddot\theta\cos\theta - mr\ell\dot\theta^2\sin\theta + 2J_p\dot\theta\dot\varphi\sin\theta\cos\theta + (J_a + J_p\sin^2\theta)\ddot\varphi = u.
\end{aligned}
$$

Physical interpretations

- Interpretation of terms
- Crosscoupling
- Orders of magnitude

Normalization

Introduce

$$\omega_0 = \sqrt{\frac{mgl}{J}},\qquad \alpha = \frac{mr\ell}{J_p},\qquad \beta = \frac{mr\ell}{J_a}$$

Choose $1/\omega_0$ as time scale and $1/J_a$ as torque scale. Then the equations become

$$
\begin{aligned}
&\ddot\theta - \dot\varphi^2\sin\theta\cos\theta + \alpha\ddot\varphi\cos\theta - \sin\theta = 0\\
&\beta\ddot\theta\cos\theta - \beta\dot\theta^2\sin\theta + 2\frac{\beta}{\alpha}\dot\theta\dot\varphi\sin\theta\cos\theta + \Big(1 + \frac{\beta}{\alpha}\sin^2\theta\Big)\ddot\varphi = u.
\end{aligned}
$$

Comparison with Pendulum on a Cart

Furuta pendulum

$$
\begin{aligned}
&\ddot\theta - \dot\varphi^2\sin\theta\cos\theta + \alpha\ddot\varphi\cos\theta - \sin\theta = 0\\
&\beta\ddot\theta\cos\theta - \beta\dot\theta^2\sin\theta + 2\frac{\beta}{\alpha}\dot\theta\dot\varphi\sin\theta\cos\theta + \Big(1 + \frac{\beta}{\alpha}\sin^2\theta\Big)\ddot\varphi = u.
\end{aligned}
$$

Pendulum on a cart

$$
\begin{aligned}
\ddot\theta + \alpha\ddot x\cos\theta - \sin\theta &= 0\\
\beta\ddot\theta\cos\theta - \beta\dot\theta^2\sin\theta + \ddot x &= u
\end{aligned}
$$

- When are the systems essentially the same?
- When do they differ significantly?

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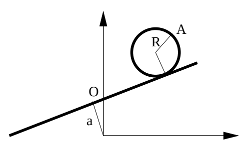
The Ball and Beam

$$
\begin{aligned}
x &= -(a + R)\sin\varphi + R\theta\cos\varphi\\
y &= (a + R)\cos\varphi + R\theta\sin\varphi
\end{aligned}
$$



Kinetic energy

$$
\begin{aligned}
2T &= m(\dot x^2 + \dot y^2) + J(\dot\varphi - \dot\theta)^2 + J_{beam}\dot\varphi^2\\
&= \Big(J + J_{beam} + m(a + R)^2 + mR^2\theta\Big)\dot\varphi^2\\
&\quad - 2\Big(J + mR(a + R)\Big)\dot\varphi\dot\theta + \Big(J + mR^2\Big)\dot\theta^2
\end{aligned}
$$

Potential energy

$$V = mg\Big((a + R)\cos\varphi + R\theta\sin\varphi\Big) + amg\cos\varphi$$

Equations of Motion

$$\frac{\partial L}{\partial \dot{\varphi}} = \Big(J + J_{beam} + m(a+R)^2 + mR^2\theta\Big)\dot{\varphi} - \Big(J + mR(a+R)\Big)$$
$$\frac{\partial L}{\partial \dot{\theta}} = \Big(J + mR(a+R)\Big)\dot{\theta} - \Big(J + mR(a+R)\Big)\dot{\varphi}$$
$$\frac{\partial L}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$$
$$\frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

Equations of motion

$$\Big(J + J_{beam} + m(a+R)^2 + mR^2\theta\Big)\ddot{\varphi} - \Big(J + mR(a+R\Big)\ddot{\theta} + \frac{1}{2}mR^2\dot{\theta}\dot{\varphi}^2 - ... = M$$
$$-\Big(J + mR(a+R)\Big)\ddot{\varphi} + \Big(J + mR(a+R)\Big)\ddot{\theta} + ... = 0$$

Mechanical Systems

Summary

- Nice systematic formalism
- Details messy
- A good case for computerized tools
- Physical interpretations
- Normalization and scaling