Exercise for Optimal control – Week 4

Choose **one** problem to solve.

Disclaimer

This is not a complete solution manual. For some of the exercises, we provide only partial answers, especially those involving numerical problems. If one is willing to use the solution manual, one should judge whether the solutions are correct or wrong by him/herself.

Exercise 1. Use tent method to derive the KKT condition (google it if you don't know) for the nonlinear optimization problem:

$$\min f(x)$$

subject to

$$\begin{aligned} g_i(x) &\leq 0, \quad i = 1, \cdots, m \\ h_j(x) &= 0, \quad j = 1, \cdots, l \end{aligned}$$

where f, g_i, h_j are continuously differentiable real-valued functions on $\mathbb{R}^n$.

Solution. (Assume that ∇f is non vanishing at the minimizer) To solve this problem, let x_* be a minimizer and define

$$\begin{aligned} \Omega_i &= \{x : g_i(x) \leq 0\}, \quad i = 1, \cdots, m \\ \Xi_j &= \{x : h_j(x) = 0\}, \quad j = 1, \cdots, l \\ \Theta &= \{x : f(x) \leq f(x_*)\} \cup \{x_*\} \end{aligned}$$

then

$$\Sigma := \bigcap_i \Omega_i \bigcap_j \Xi_j \bigcap \Theta = \{x_*\}.$$

The tents of the defined sets are

$$\begin{aligned} K^{\Omega_i} &= \begin{cases} \{x : \nabla g_i(x_*)(x - x_*) \leq 0\}, & x_* \in \partial\Omega_i \\ \mathbb{R}^n, & x_* \in \mathrm{Int}\Omega_i \end{cases} \\ K^{\Xi^j} &= \{x : \nabla h_j(x_*)(x - x_*) = 0\} \\ K^{\Theta} &= \{x : \nabla f(x_*)(x - x_*) \leq 0\} \end{aligned}$$

By Lemma 2 in the lecture notes, these tents are separable and there exist vectors ω_i, ξ_i, θ satisfying

$$\begin{aligned} \omega_i^\top (x - x_*) &\leq 0, \quad \forall x \in K^{\Omega_i} \\ \xi_j^\top (x - x_*) &\leq 0, \quad \forall x \in K^{\Xi_j} \\ \theta^\top (x - x_*) &\leq 0, \quad \forall x \in K^{\Theta} \end{aligned}$$

and

$$\sum_i \omega_i + \sum_j \xi_j + \theta = 0 \tag{1}$$

Since K^Θ is not its affine hull (a half space in fact). There there exist $\mu_i \geq 0$, $\theta_0 \geq 0$ and ν_j – signs undetermined – such that

$$\omega_i = \mu_i \nabla g_i(x_*), \quad \theta = \theta_0 \nabla f(x_*), \quad \xi_j = \nu_j \nabla h_j(x_*).$$

(Note that K^{Ξ^j} is the tangent space of $\{h(x) = 0\}$, thus ξ_j must be orthogonal to the tangent space, which must be aligned with the gradient of ∇h_j) Plugging into (1), we get the KKT condition:

$$\theta_0 \nabla f(x_*) + \sum_i \mu_i \nabla g_i(x_*) + \sum_j \nu_j \nabla h_j(x_*) = 0 \tag{2}$$

for constants $\theta_0 \geq 0$, $\mu_i \geq 0$. On the other hand, if $g_i(x_*) < 0$, then μ_i must be zero. This is equivalent to saying

$$\sum_i \mu_i g_i(x_*) = 0. \tag{3}$$

Conditions (2) and (3) together is the KKT condition.

Exercise 2. Find a variation of inputs u_ϵ near u_* that generate the deviation vectors of the form $\sum_{i=1}^q k_i v_i(t_f)$ for $k_i \geq 0$, where $v_i(t_f)$ is generated by

$$u_{i,\epsilon}(t) = \begin{cases} w_i, & t \in (\tau_i - \epsilon, \tau_i] \\ u_*(t), & \text{otherwise} \end{cases} \tag{4}$$

See lecture note. *Hint*: consider the combined needle variation

$$u_\varepsilon(t) = \left\{ \begin{array}{ll} w_i, & t \in (\tau_i - k_i\varepsilon, \tau_i] \text{ for some } i \in \{1, \cdots, q\} \\ u_*(t), & \text{otherwise} \end{array} \right. .$$

Then find $x_\epsilon(t_f)$ and $\frac{\partial x_\epsilon(t_f)}{\partial t}|_{\epsilon=0+}$. Start with $q = 2$.

Solution. It suffices to show for $q = 1$ and $q = 2$ and then conclude by induction.

Let us denote $x(t, u)$ as the solution to $\dot{x} = f(x, u)$ at t under control input u. For notation ease, denote

$$u_1 = \begin{cases} w_1, & t \in (\tau_1 - k_1\epsilon, \tau_1] \\ u_*(t), & \text{otherwise} \end{cases}, \quad u_2 = \begin{cases} w_1, & t \in (\tau_1 - k_1\epsilon, \tau_1] \\ w_2, & t \in (\tau_2 - k_2\epsilon, \tau_2] \\ u_*(t), & \text{otherwise} \end{cases}$$

One should be aware that u_1 and u_2 are functions of ϵ. When $q = 1$, by integrating the system, we get

$$\begin{aligned} x(\tau_1, u_1) &= x_*(\tau_1 - k_1\epsilon, u_*) + \int_{\tau_1 - k_1\epsilon}^{\tau_1} f(x(s, u_1), w)\mathrm{d}s, \\ &= x_*(\tau_1 - k_1\epsilon, u_*) + \int_{\tau_1 - k_1\epsilon}^{\tau_1} f(x_*(s), u_*(s))\mathrm{d}s + \int_{\tau - \epsilon}^{\tau} [f(x(s, u_1), w) - f(x_*(s), u_*(s))]\mathrm{d}s \\ &= x_*(\tau_1) + \int_{\tau_1 - k_1\epsilon}^{\tau_1} [f(x(s, u_1), w) - f(x_*(s), u_*(s))]\mathrm{d}s \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial x(\tau_1, u_1)}{\partial \epsilon}|_{0+} &= \lim_{\epsilon \to 0+} \frac{x(\tau_1, u_1) - x_*(\tau_1)}{\epsilon} \\ &= \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_{\tau_1 - k_1\epsilon}^{\tau_1} [f(x(s, u_1), w) - f(x_*(s), u_*(s))]\mathrm{d}s \\ &= k_1[f(x_*(\tau_1), w) - f(x_*(\tau_1), u_*(\tau_1))] \\ &= k_1 v_1(\tau_1) \end{aligned}$$

Hence, the deviation vector is k_1 times the deviation vector v_1 obtained under needle variation $u_{1,\epsilon}$ defined in (4). Thus we see if $v_1(t_f)$ is a deviation vector, so is $k_1 v_1(t_f)$ for any $k_1 > 0$.

Now, consider $q = 2$. We need to find

$$v(t_f) = \frac{\partial x(t_f, u_2)}{\partial \epsilon}|_{0+} = \lim_{\epsilon \to 0+} \frac{x(t_f, u_2) - x_*(t_f)}{\epsilon}.$$

For that, we first find

$$v(t) = \lim_{\epsilon \to 0+} \frac{x(t, u_2) - x_*(t)}{\epsilon}$$

for $t \geq \tau_2$, as in the needle variation case. To find $x(t, u_2)$, we must know $x(\tau_2, u_2)$, which is

$$x(\tau_2, u_2) = x(\tau_2 - k_2\epsilon, u_1) + \int_{\tau_2 - k_2\epsilon}^{\tau_2} f(x(s, u_2), w_2) \mathrm{d}s$$

Note that

$$\lim_{\epsilon \to 0+} \frac{x(t, u_2) - x_*(t)}{\epsilon} = \lim_{\epsilon \to 0+} \frac{x(t, u_2) - x(t, u_1)}{\epsilon} + \frac{x(t, u_1) - x_*(t)}{\epsilon}.$$

The second term on the RHS is nothing but $k_1 v_1(t)$ since $t \geq \tau_2 > \tau_1$. It suffices to show

$$\lim_{\epsilon \to 0+} \frac{x(t, u_2) - x(t, u_1)}{\epsilon} = k_2 v_2(t)$$

where $v_2(t)$ (at $t = t_f$) corresponds to the deviation vector under needle variation $u_{2,\epsilon}$ defined as (4). It is sufficient to verify for $t = \tau_2$. To that end, we calculate

$$x(\tau_2, u_2) = x(\tau_2 - k_2\epsilon, u_1) + \int_{\tau_2 - k_2\epsilon}^{\tau_2} f(x(s, u_2), w_2) \mathrm{d}s$$

(note that $x(\tau_2 - k_2\epsilon, u_2) = x(\tau_2 - k_2\epsilon, u_1)$), and

$$x(\tau_2, u_1) = x(\tau_2 - k_2\epsilon, u_1) + \int_{\tau_2 - k_2\epsilon}^{\tau_2} f(x(s, u_1), u_*(s)) \mathrm{d}s.$$

It follows that

$$\begin{aligned} \frac{x(\tau_2, u_2) - x(\tau_2, u_1)}{\epsilon} &= \frac{1}{\epsilon} \int_{\tau_2 - k_2\epsilon}^{\tau_2} f(x(s, u_2), w_2) - f(x(s, u_1), u_*(s)) \mathrm{d}s \\ &\to k_2[f(x_*(\tau_2), w_2) - f(x_*(\tau_2), u_*(\tau_2))] \\ &= k_2 v_2(\tau_2) \end{aligned}$$

as desired.

Exercise 3. Consider driving a cart (a unicycle, or Dubins car) on the plane

$$\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\theta} &= \omega \end{aligned}$$

where (x, y) represents the position of the cart and θ the heading angle, the driving speed is a constant $v > 0$, see Figure 1. There is only one control: the turning rate ω, which is bounded by

$$|\omega| \leq \frac{v}{R}$$

for some positive constant R. Study the time optimal control problem of driving the cart from initial position at

$$(x(0), y(0), \theta(0))^\top = (0, 0, 0)^\top$$

to

$$(x(t_f), y(t_f), \theta(t_f))^\top = (x_f, y_f, \theta_f)^\top \in \mathbb{R}^3$$

What are the possible types of trajectories joining the initial and terminal states? *Note 1: there may exist singular arcs! Check lecture note 4. Note 2: there may exist several solutions to the maximum principle.*

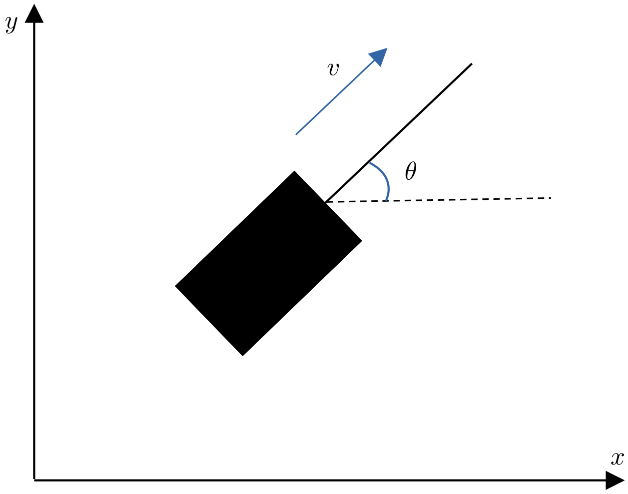

Figure 1: A unicycle.

Solution. The cost for this problem is

$$J = \int_0^{t_f} 1 \mathrm{d}t.$$

The Hamiltonian function is $H = p_1 v \cos\theta + p_2 v \sin\theta + p_3 \omega + p_0$, and the costate equation is

$$\begin{aligned} \dot{p}_1 &= \dot{p}_2 = 0 \\ \dot{p}_3 &= p_1 v \sin\theta - p_2 v \cos\theta \end{aligned}$$

The maximum principle gives

$$\omega(t) = \begin{cases} \frac{v}{R}, & p_3^*(t) < 0 \\ -\frac{v}{R}, & p_3^*(t) > 0 \\ ?, & p_3^*(t) = 0 \end{cases}.$$

We need to check if there is singular arc. Suppose that $p_3^*(t) = 0$ for $t \in [t_1, t_2]$, we must have $\dot{p}_3^*(t) = 0$ on $[t_1, t_2]$, or

$$p_1 \sin\theta - p_2 \cos\theta = 0,$$

which happens only when $\theta^*(t)$ is a constant on $[t_1, t_2]$ or $p_1 = p_2 = 0$ – the latter case cannot be true. Thus on these intervals $\omega^* = 0$.

Thus, on the optimal path, there are at most three types of driving: turn left or right with the maximum rate, or go straight. At this stage, the MP may give several solutions. Numerically, we can solve the problem by boundary value problem solver by choosing different initial conditions and then choose the one with the minimum cost. In [1], the authors showed that there are only two types of optimal solutions:

1) $B_a S_b B_c$ where B is either r or l, and a, c are the driving time within $[0, \frac{2\pi R}{v})$ units of time. S_b means go straight for b units of time.

2) $B_a B_b B_c$, either $r_a l_b r_c$ or $l_a r_b l_c$, with $b \in (\frac{\pi R}{v}, \frac{2\pi R}{v})$, $\min\{a, c\} < b - \frac{\pi R}{v}$ and $\max\{a, c\} < b$.

References

[1] Héctor J Sussmann and Guoqing Tang. Shortest paths for the Reeds-Shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control. *Rutgers Center for Systems and Control Technical Report*, 10:1–71, 1991.