Exercise for Optimal control – Week 3

Choose **1.5** problems to solve.

Disclaimer

This is not a complete solution manual. For some of the exercises, we provide only partial answers, especially those involving numerical problems. If one is willing to use the solution manual, one should judge whether the solutions are correct or wrong by him/herself.

Exercise 1. Consider a harmonic oscillator $\ddot{x} + x = u$ whose control is constrained in the interval $[-1, 1]$. Find an optimal controller u which drives the system at initial state $(x(0), \dot{x}(0)) = (X_1, X_2)$ to the origin in minimal time. Draw the phase plot.

Solution. Rewrite the system in standard form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u$$

The cost for this problem is

$$J = \int_0^{t_f} 1 \mathrm{d}t$$

where t_f is free. The Hamiltonian for the system is

$$H = p_1 x_2 + p_2(-x_1 + u) + p_0$$

and the costate equation is

$$\dot{p}_1 = p_2$$
$$\dot{p}_2 = -p_1$$

Note that (p_1, p_2) is also a harmonic oscillator – $\frac{d}{dt}(p_1^2 + p_2^2) = 0$ – we can solve them as

$$p_1 = r \sin(t + \phi)$$
$$p_2 = r \cos(t + \phi)$$

for some constants $r > 0$, $\phi \in (-\pi, \pi)$. The maximum principle says

$$u^*(t) = \mathrm{sign}(p_2^*(t)).$$

Thus the dwell time for each switch is exactly π seconds. Except at the switching point, $u^*(t)$ is constant, and we can find the trajectory of $(x_1^*(t), x_2^*(t))$: when $u^* = 1$,

$$(x_2 - 1)^2 + x_1^2 = \mathrm{const}$$

since

$$\frac{d}{dt}[(x_1 - 1)^2 + x_2^2] = 2(x_1 - 1)x_2 + 2x_2(-x_1 + 1) = 0$$

Consequently $x_2(t) = 1 + r\sin(t + \theta_0)$, $x_1 = r\cos(t + \theta_0)$, and the system moves clockwisely on the circle. Similarly, when $u^* = -1$,

$$(x_2 + 1)^2 + x_1^2 = \mathrm{const}$$

Thus we can draw the phase plot of the system.

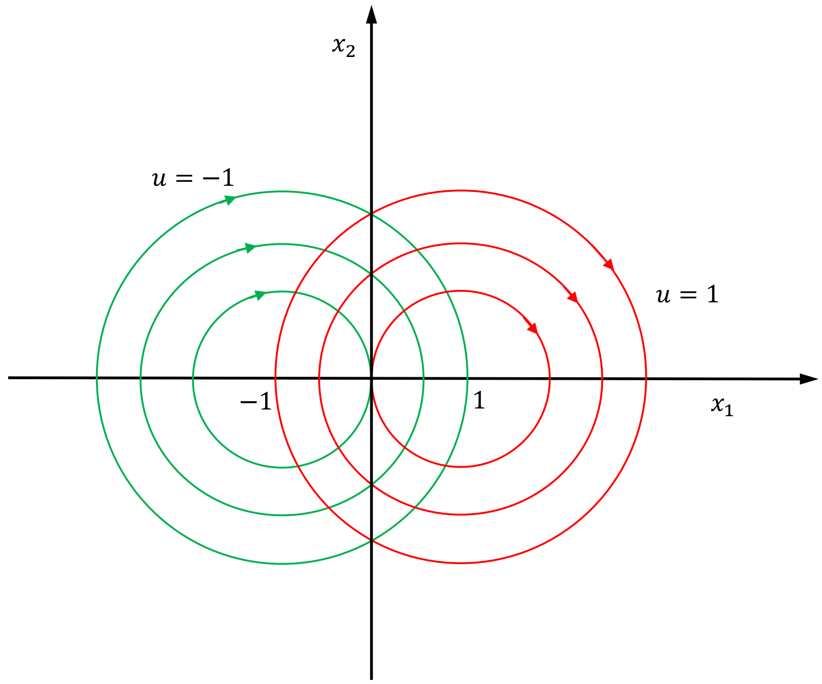

Figure 1: The phase plot for $u = \pm 1$.

The red color circles have center at $(1, 0)$ and the greens at $(-1, 0)$. Let us trace the system trajectory backward from $t = t_f$. At the final stage, in order to reach the origin, only two arcs are possible – note that on each circle, the system can travel at most π seconds.) see Figure 2.

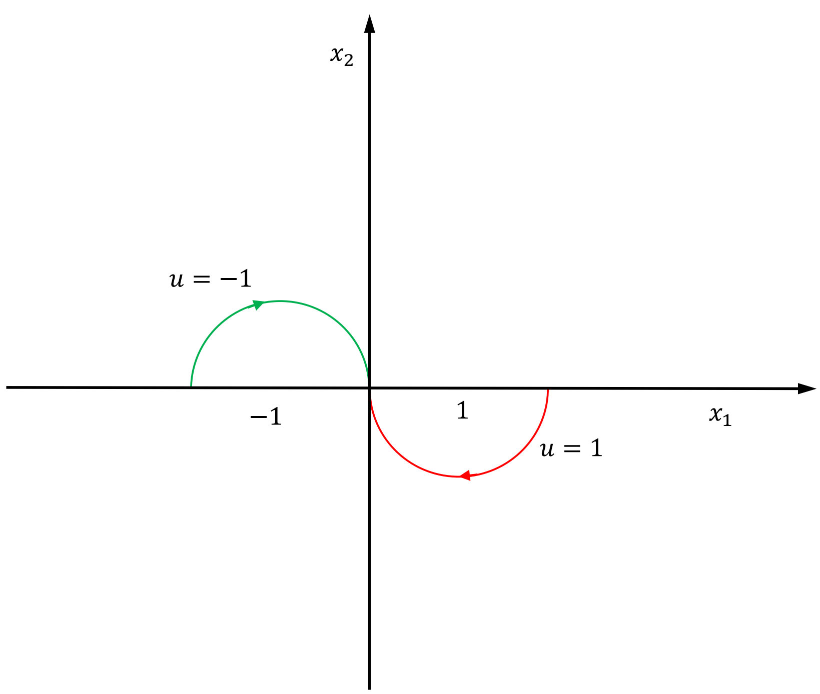

Figure 2: The phase plot at the final stage.

To find the previous arc, choose a point A as in Figure 3, then draw a line passing through A and

$(-1, 0)$. The intersection of this dashed line with the circle determined by A and center $(-1, 0)$ is denoted A', which lies on the circle $(x_1+3)^2+x_2^2 = C^2$. Thus for all initial states on the arch between A' and A, they should flow along the arch and then reach point A and goes to zero following the final stage arc. To find the trajectory before A', one goes backward along the red circles. More precisely, draw a line passing A' and $(1, 0)$, which will intersect the half circle $\{(x_1, x_2) : (x_1 - 5)^2 + x_2^2 = 1\}$ at some point A'', then the trajectory before A' should lie on the arc $\widehat{A'A''}$. Continuing this procedure, we can find the optimal trajectory for all for arbitrary initial condition. However, an analytic solution is not obvious. Theoretically, fixing A, one can compute the complete trajectory starting from starting from $t = -\infty$. These trajectories will span the whole state space. Thus it suffices to determine which trajectory it lies on.

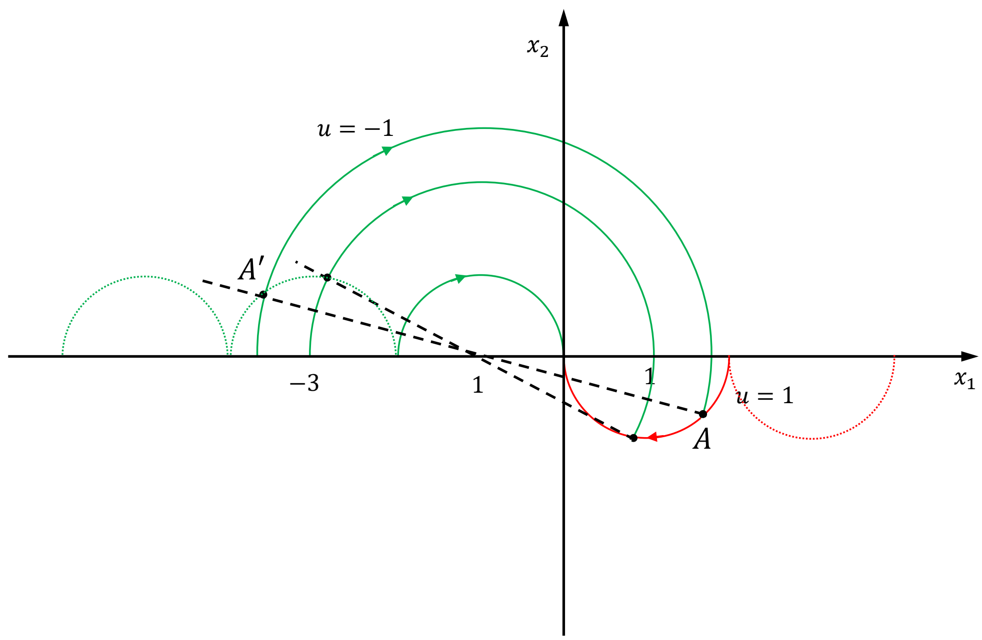

Figure 3: The phase plot of the last two stages.

Exercise 2. Consider a rocket, modeled as a particle of constant mass m moving in zero gravity empty space. Let $u \geq 0$ be the mass flow, assumed to be a known function of time, let c be the constant thrust velocity and v an angle that can be controlled. See Figure 4. The equations of motion are

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 &= \frac{c}{m}u(t)\cos(v(t))\\ \dot{x}_4 &= \frac{c}{m}u(t)\sin(v(t))\end{aligned}$$

1) Show that cost functionals of the class

$$\min_{v(\cdot)} \int_0^{t_f} \mathrm{d}t \text{ or } \min_{v(\cdot)} \phi(x(t_f))$$

gives the optimal control

$$\tan v^*(t) = \frac{c_1 + c_2 t}{c_3 + c_4 t}.$$

2) Assume that the rocket starts at rest at the origin and that we want to drive it to a given height $x_2(t_f)$ in a given time t_f such that the final velocity in the horizontal direction $x_3(t_f)$ is maximized while $x_4(t_f) = 0$. Show that the optimal control is reduced to a linear tangent law

$$\tan v^*(t) = c_1 + c_2 t.$$

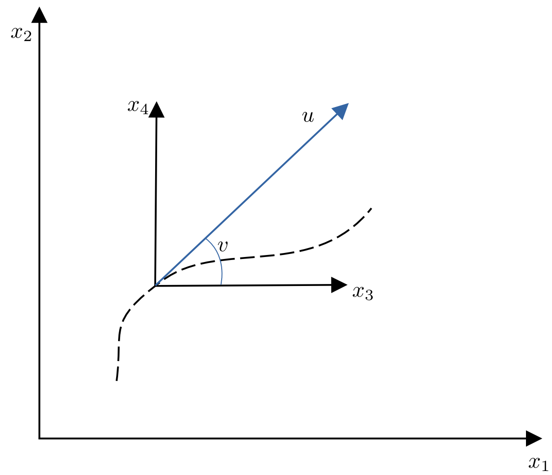

Figure 4: A rocket model.

3) Let the rocket in represent a missile whose target is at rest. Minimize the transfer time $t_f$from the state $[0, 0, x_3(0), x_4(0)]$ to the state $[x_1(t_f), x_2(t_f), \text{free}, \text{free}]$. Solve the problem under the assumption that u is constant.

4) To increase the realism now assume that the motion is under a constant gravitational force. The only equation that needs to be modified is the one for x_4 (the acceleration in the vertical direction):

$$\dot{x}_4 = \frac{c}{m} u(t) \sin(v(t)) - g.$$

Show that the optimal law is still optimal for the cost functional

$$\min_{v(\cdot)} \phi(x(t_f)) + \int_0^{t_f} \mathrm{d}t.$$

5) Now we take into consideration of the mass loss of the rocket. Let x_5 denote the mass of the rocket. The overall equations of motion now read

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{c}{m} u(t) \cos(v(t)) \\
\dot{x}_4 &= \frac{c}{m} u(t) \sin(v(t)) - g \\
\dot{x}_5 &= -u(t)
\end{aligned}$$

where $u \in [0, u_{\max}]$. Show that the optimal solution to transferring the rocket from a state of given position, velocity and mass to a given altitude $x_2(t_f)$ using a given amount of fuel, such that the distance $x_1(t_f) - x_1(0)$ is maximized, is

$$v^*(t) = \text{constant}, \; u^*(t) = \{u_{\max}, 0\}.$$

Solution. 1) First, for the time optimal case, the Hamiltonian is $H = p_1 x_3 + p_2 x_4 + p_3 \frac{c}{m} u(t) \cos v + p_4 \frac{c}{m} u(t) \sin(v) + p_0$. The costate equation is

$$\begin{aligned}
\dot{p}_1 &= 0 \\
\dot{p}_2 &= 0 \\
\dot{p}_3 &= -p_1 \\
\dot{p}_4 &= -p_2
\end{aligned}$$

from which we see

$$\begin{aligned} p_1^* &= c_1, \\ p_2^* &= c_2, \\ p_3^*(t) &= -c_1 t + c_3 \\ p_4^*(t) &= -c_2 t + c_4 \end{aligned}$$

First, we shall notice that p_3 and p_4 cannot be zero at the same time, otherwise, $p_1 = p_2 = p_0 = 0$, a contradiction. To maximize H w.r.t. v, we write H as

$$H = \frac{c}{m} u(t) \sqrt{p_3^2 + p_4^2} \sin(v + \alpha) + *$$

for some

$$\tan \alpha(t) = \frac{p_3^*(t)}{p_4^*(t)} \in [-\pi, \pi)$$

where $*$ represent some terms not dependent on v. Hence v should be taken as

$$v^*(t) = -\alpha(t) + \frac{\pi}{2}$$

where α depends on the costate (p_3, p_4). Now

$$\tan v^*(t) = \tan(-\alpha(t) + \frac{\pi}{2}) = \frac{p_4^*(t)}{p_3^*(t)} = \frac{-c_2 t + c_4}{-c_1 t + c_3}. \tag{1}$$

For the second case, the Hamiltonian differs with the time optimal one only by a constant p_0, thus the maximum principle is the same. Therefore, the form of the optimal controller doesn't change.

2) First we need to formulate the problem:

$$\min_v -x_3(t_f)$$

subject to

$$x_2(t_f) = x_{2f}, \quad x_4(t_f) = 0.$$

The Hamiltonian is the same as in the second case of 1) and thus the optimal controller should satisfy

$$\tan v^*(t) = \frac{-c_2 t + c_4}{-c_1 t + c_3}.$$

But now we have some extra constraints and we may possibly solve for some constants. Since $x_2(t_f)$ and $x_4(t_f) = 0$ is fixed, we must have

$$\begin{bmatrix} p_1^*(t_f) \\ p_2^*(t_f) \\ p_3^*(t_f) \\ p_4^*(t_f) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} v_1 \\ 0 \\ v_2 \\ 0 \end{bmatrix}$$

where v_1, v_2 are free in $\mathbb{R}$. Thus $p_1^*(t_f) = 0$, $p_3^*(t_f) = 1$. From $p_1^*(t_f) = 0$ we immediately get $c_1 = 0$.

3) The problem is

$$\min \int_0^{t_f} 1 dt$$

subject to

$$\begin{aligned} x(0) &= [0, 0, x_{3f}, x_{4f}]^\top \\ x(t_f) &= [x_{1f}, x_{2f}, \text{free}, \text{free}]^\top \end{aligned}$$

The Hamiltonian is the same as in the first case of 1), thus the controller is of the form (1). As in 2), we can get $p_3^*(t_f) = p_4^*(t_f) = 0$. Thus $p_3(t) = -c_1(t - t_f)$, $p_4(t) = -c_2(t - t_f)$ from which it follows that

$$\tan v^*(t) = \frac{c_2}{c_1}.$$

In other words, $v^*(t)$ is constant. Now that u is constant, we can solve the system equation

$$\begin{aligned} x_3(t) &= \frac{c}{m}ut\cos(v^*) + x_{3i} \\ x_4(t) &= \frac{c}{m}ut\sin(v^*) + x_{4i} \\ x_1(t) &= \frac{1}{2}\frac{c}{m}ut^2\cos(v^*) + +x_{3i}t \\ x_2(t) &= \frac{1}{2}\frac{c}{m}ut^2\sin(v^*) + x_{4i}t \end{aligned}$$

Putting $t = t_f$ in the last two lines, we get

$$\begin{aligned} \frac{1}{2}\frac{c}{m}ut_f^2\cos(v^*) + x_{3i}t_f &= x_{1f} \\ \frac{1}{2}\frac{c}{m}ut_f^2\sin(v^*) + x_{4i}t_f &= x_{2f} \end{aligned}$$

from which we can find t_f and v^*.

4) The new Hamiltonian is $H = p_1x_3 + p_2x_4 + p_3\frac{c}{m}u(t)\cos v + p_4(\frac{c}{m}u(t)\sin(v) - g) + p_0$. Since the new term p_4g does not depend on u, the MP is the same.

5) The problem is

$$\min -x_1(t_f)$$

subject to

$$x(t_f) = [\text{free}, x_{2f}, \text{free}, \text{free}, x_{5f}]^\top.$$

And here u is also a control input. The Hamiltonian is

$$\begin{aligned} H &= p_1x_3 + p_2x_4 + p_3\frac{c}{x_5}u(t)\cos v + p_4(\frac{c}{x_5}u(t)\sin(v) - g) - p_5u(t) \\ &= (p_3\frac{c}{x_5}\cos(v) + p_4\frac{c}{x_5}\sin(v) - p_5)u(t) + * \end{aligned}$$

Since $u(t) \geq 0$, we may have two different cases. Note that $x_5 > 0$.

Case 1. $u = 0$: In this case, controller has no effect on the cost, then v can be taken arbitrarily.

Case 2. If $u > 0$: For this case, the maximum principle is the same as the second case of 1). Hence the optimal controller is in the form (1). However, we have an extra component in the costate equation,

$$\dot{p}_5 = p_3\frac{c}{x_5^2}u(t)\cos v + p_4\frac{c}{x_5^2}u(t)\sin v \tag{2}$$

The terminal constraint tells us that

$$\begin{bmatrix} p_1^*(t_f) \\ p_2^*(t_f) \\ p_3^*(t_f) \\ p_4^*(t_f) \\ p_5^*(t_f) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \perp \begin{bmatrix} v_1 \\ 0 \\ v_3 \\ v_4 \\ 0 \end{bmatrix}$$

for v_1, v_3, v_4 free. Thus

$$\begin{aligned} p_1^*(t_f) &= 1, \\ p_3^*(t_f) &= 0, \\ p_4^*(t_f) &= 0. \end{aligned}$$

Thus $p_1^* = c_1 = 1$, and $p_3^*(t) = -(t - t_f)$, $p_4^*(t) = -c_2(t - t_f)$ and

$$\tan v^*(t) = c_2 = \text{const}.$$

The input u is yet to be determined. To maximize H,

$$u^*(t) = \begin{cases} 0 & \sigma(t) < 0 \\ u_{\max} & \sigma(t) > 0 \\ ? & \sigma(t) = 0. \end{cases}$$

where

$$\sigma(t) = p_3 \frac{c}{x_5} \cos(v) + p_4 \frac{c}{x_5} \sin(v) - p_5.$$

One can then show that

$$\begin{aligned}
\dot{\sigma}(t) &= \dot{p}_3(t) \frac{c}{x_5} \cos(v) - p_3 \frac{c\dot{x}_5}{x_5^2} \cos v + \dot{p}_4 \frac{c}{x_5} \sin(v) - p_4 \frac{c\dot{x}_5}{x_5^2} \sin(v) - \dot{p}_5 \\
&= \dot{p}_3(t) \frac{c}{x_5} \cos(v) + \dot{p}_4 \frac{c}{x_5} \sin(v) + p_3 \frac{cu}{x_5^2} \cos v + p_4 \frac{cu}{x_5^2} \sin(v) - \dot{p}_5 \\
&= \dot{p}_3(t) \frac{c}{x_5} \cos(v) + \dot{p}_4 \frac{c}{x_5} \sin(v) \text{ (see (2))} \\
&= \frac{c}{x_5} (\dot{p}_3(t) \cos(v) + \dot{p}_4 \sin(v)) \\
&= \frac{c}{x_5} (-\cos(v) - c_2 \sin(v))
\end{aligned}$$

Now recall that $\tan v = c_2$, or $\sin(v) = c_2 \cos(v)$, the above can be further written as

$$\dot{\sigma}(t) = \frac{c}{x_5} (-\cos(v) - c_2^2 \cos(v)) = -\frac{c(1 + c_2^2)}{x_5(t)} \cos(v) < 0$$

Thus σ is decreasing and there is at most one switch.

Exercise 3. Try to solve the Rayleigh problem: consider minimizing

$$J = \int_0^{t_f} (u^2 + x_1^2) \mathrm{d}t$$

subject to (the controlled van de Pol oscillator):

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u
\end{aligned}$$

with initial condition $(x_1(0), x_2(0)) = (-5, -5)$, $t_f = 4.5$ and a mixed input and state constraint:

$$-1 \le u(t) + \frac{x_1(t)}{6} \le 0.$$

Draw the optimal controller and the state trajectory. You may use numerical methods, e.g., discretization.

Solution. (Courtesy by Manu) The Hamiltonian is

$$H = p_1 x_2 + p_2(-x_1 + x_2(1.4 - 0.14x_2^2) + 4u) - (u^2 + x_1^2)$$

and the costate equation is

$$\begin{aligned}
\dot{p}_1 &= p_2 + 2x_1 \\
\dot{p}_2 &= -p_1 - p_2(1.4 - 0.42x_2^2)
\end{aligned}$$

with boundary condition $p_1(t_f) = p_2(t_f) = 0$. The maximum principle gives

$$u^*(t) = \begin{cases} 2p_2^*(t), & -6 \le 12p_2^*(t) + x_1^*(t) \le 0 \\ -x_1^*(t)/6, & 0 < 12p_2^*(t) + x_1^*(t) \\ -1 - x_1^*(t)/6, & 12p_2^*(t) + x_1^*(t) < -6. \end{cases}$$

The problem of solving for the optimal state trajectory and the costates is a two-point boundary value problem. The code below numerically solves the problem.

```
import numpy as np
from scipy.integrate import solve_bvp
import matplotlib.pyplot as plt
# https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_bvp.html
def u_star(x_1,p_2):
 if 0 < 12*p_2 + x_1:
  return -x_1/6
 elif 12*p_2 + x_1 < -6:
  return -1-x_1/6
 else:
  return 2*p_2

def fun(t, xp):
 x_1 = xp[0]
 x_2 = xp[1]
 p_1 = xp[2]
 p_2 = xp[3]
 u = np.array(list(map(u_star,x_1,p_2)))
 x_1_dot = x_2
 x_2_dot = -x_1+x_2*(1.4-0.14*x_2**2)+4*u
 p_1_dot = p_2 + 2*x_1
 p_2_dot = -p_1-p_2*(1.4-0.42*x_2**2)
 return np.vstack((x_1_dot, x_2_dot, p_1_dot, p_2_dot))

def bc(xp_0, xp_tf):
 x_1_0 = xp_0[0]
 x_2_0 = xp_0[1]
 p_1_0 = xp_0[2]
 p_2_0 = xp_0[3]
 x_1_tf = xp_tf[0]
 x_2_tf = xp_tf[1]
 p_1_tf = xp_tf[2]
 p_2_tf = xp_tf[3]
 return np.array([x_1_0+5,x_2_0+5,p_1_tf,p_2_tf])

t_f = 4.5
N = 100
ts = np.linspace(0, t_f, N+1) # Initial mesh
xp_init = np.zeros((4, ts.size)) # Initial guess
sol = solve_bvp(fun, bc, ts, xp_init, tol = 1e-9, verbose = 2)

ts = np.linspace(0, t_f, 1000)
xp = sol.sol(ts)
x_1 = xp[0]
x_2 = xp[1]
p_1 = xp[2]
p_2 = xp[3]
plt.figure(0)
plt.plot(x_1,x_2,label="$(x_1,x_2)$")
plt.legend()
plt.xlabel(r"$x_1$")
plt.ylabel(r"$x_2$")
plt.savefig('states.png', dpi=400)

u = np.array(list(map(u_star,x_1,p_2)))
plt.figure(1)
```

```
plt.plot(ts,u,label="$u^{\star}$")
plt.legend()
plt.xlabel(r"$t$")
plt.ylabel(r"$u^{\star}$")
plt.savefig('control.png', dpi=400)
```

The results is shown below:

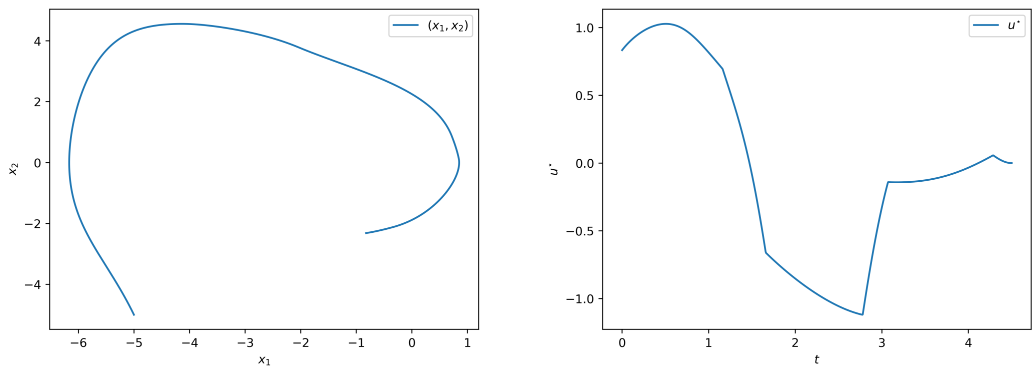

Figure 5: Optimal solution.