Exercise for Optimal control – Week 2

Choose **one** problem to solve.

Disclaimer

This is not a complete solution manual. For some of the exercises, we provide only partial answers, especially those involving numerical problems. If one is willing to use the solution manual, one should judge whether the solutions are correct or wrong by him/herself.

Exercise 1 (Insect control)**.** Let $w(t)$ and $r(t)$ denote, respectively, the worker and reproductive population levels in a colony of insects, e.g. wasps. At any time t, $0 \leq t \leq T$ in the season the colony can devote a fraction $u(t)$ of its effort to enlarging the worker force and the remaining fraction $u(t)$ to producing reproductives. The per capita mortality rate of workers is μ and the per capita natality rate is b when full effort is put on the worker population. Assume $\mu < b$. The two populations are governed by the equations

$$\begin{aligned}\dot{w} &= (bu - \mu)w \\ \dot{r} &= c(1-u)w\end{aligned}$$

with $(w(0), r(0) = (1, 0)$, where u satisfies the constraint $0 \leq u(t) \leq 1$. The objective is to maximize $r(T)$ or minimize

$$J = -r(T).$$

Solution. Since $L = 0$, the Hamiltonian for this problem is $H = p_1(bu - \mu)w + p_2c(1-u)w$. The costate equation reads

$$\begin{aligned}\dot{p}_1 &= -p_1(bu - \mu) - p_2c(1-u) \\ \dot{p}_2 &= 0\end{aligned}$$

with terminal condition $p_1(T) = 0$, $p_2(T) = 1$. Thus $p_2(t) \equiv 1$ and

$$H = (p_1b - c)wu + (c - p_1\mu)w.$$

Since $w > 0$ for all $t \geq 0$, the optimal control law is

$$u^*(t) = \begin{cases} 1, & p_1(t)b \geq c \\ 0, & p_1(t)b < c \end{cases}.$$

Since $p_1(T) = 0$, then near T, u should be taken as 0. Moving backward, assume t_s is the first time instance that $p_1(t_s)b = c$. Then on $[t_s, T]$,

$$\dot{p}_1 = \mu p_1 - c$$

Solving this, we obtain

$$p_1(t) = e^{\mu(t-t_s)}p_1(t_s) + \frac{c}{\mu}(1 - e^{\mu(t-t_s)}) = \frac{c}{b}e^{\mu(t-t_s)} + \frac{c}{\mu}(1 - e^{\mu(t-t_s)})$$

(remember that $p_1(t_s)b = c$.) Now at $t = T$, $p_1(T) = 0$, or

$$0 = \frac{1}{b}e^{\mu(T-t_s)} + \frac{1}{\mu}(1 - e^{\mu(T-t_s)})$$

from which it follows that

$$t_s = T - \frac{1}{\mu}\ln\left(\frac{b}{b-\mu}\right). \tag{1}$$

Continuing moving backward, the costate equation becomes

$$\dot{p}_1 = -p_1(b - \mu)$$

with terminal condition $p_1(t_s) = \frac{c}{b} > 0$. Solving the equation, we obtain

$$p_1(t) = e^{(b-\mu)(t_s - t)} p_1(t_s), \quad t \in [0, t_s]$$

which is a decreasing function on $[0, t_s]$. This justifies the optimality of $u^*(t)$, i.e., $p(t)c > b$ on $[0, t_s]$. Hence

$$u^*(t) = \begin{cases} 1, & 0 \le t < t_s \\ 0, & t_s \le t \le T \end{cases}$$

where t_s is as (1).

Exercise 2 (Time optimal control of a lunar lander)**.** Study the time optimal control of the moon lander problem (see lecture notes for the model). ~~In addition, argue that there is an optimal feedback controller.~~

Solution. The model for the moon lander is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g + u \end{aligned}$$

where $0 < g < 1$ and $|u| \le 1$, with initial condition $(x_1(0), x_2(0)) = (h, v)$ and terminal constraint $(x_1(t_f), x_2(t_f)) = (0, 0)$. Now for time optimal control, the cost is

$$J = \int_0^{t_f} 1 \mathrm{d}t.$$

For this problem, $H = p_1 x_2 + p_2(-g + u) + p_0$ and the costate equation is

$$\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \end{aligned}$$

from which we solve $p_1 = c_1$ and $p_2(t) = -c_1 t + c_2$ for some constants c_1, c_2. The MP says $u^*(t) = \mathrm{sign}(p_2^*(t))$. Notice that it is necessary that $p_2^*(t_f) > 0$ in order to land successfully. There are two possible cases.

Case 1: $c_1 \ge 0$. In this case, there is no switch, and the controller is $u \equiv 1$, which happens only when $1 - g = \frac{v^2}{2h}$.

Case 2: $c_1 < 0$. There is at most one switch, say t_s. Then $-c_1 t_s + c_2 = 0$, which implies $c_2 < 0$. In this case, we can calculate the system trajectory as

$$\begin{aligned} x_1(t) &= h + vt - \frac{1}{2}(1 + g)t^2 \\ x_2(t) &= v - (1 + g)t \end{aligned}$$

for $t \in [0, t_s]$, and

$$\begin{aligned} x_1(t) &= h + t_s^2 + (v - 2t_s)t + \frac{1}{2}(1 - g)t^2 \\ x_2(t) &= v - 2t_s + (1 - g)t \end{aligned}$$

Put $x_1(t_f) = x_2(t_f) = 0$, we get two equations

$$\begin{aligned} v - 2t_s + (1 - g)t_f &= 0 \\ h + t_s^2 + (v - 2t_s)t_f + \frac{1}{2}(1 - g)t_f^2 &= 0 \end{aligned}$$

from which we can solve

$$t_s = \frac{v + \sqrt{v^2 - \frac{1}{2}(1+g)(v^2 - 2(1-g)h)}}{1+g}$$

Notice that $v^2 < 2(1-g)h$ – in order to have a switch, thus t_s is well-defined. To conclude

$$u^*(t) = \begin{cases} -1, & 0 \le t \le t_s \\ 1, & t > t_s \end{cases}.$$

Thus the lander first accelerate and then de-accelerate – unlike the minimum fuel control, there is no free falling process.

Exercise 3 (Minimum fuel and time control)**.** Consider the planar system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

where u satisfies the constraint $|u(t)| \le 1$ for all $t \in [0, t_f]$. For any given initial state (ξ_1, ξ_2), find an optimal control u_* which drives the state to $(0,0)$ while minimizing

$$J = t_f + \int_0^{t_f} |u(t)|\mathrm{d}t = \int_0^{t_f} (1 + |u(t)|\mathrm{d}t.$$

Solution. The Hamiltonian is $H = p_1 x_2 + p_2 u + p_0(1 + |u|)$ and the costate equation

$$\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \end{aligned}$$

from which we get $p_1 = c_1$, $p_2(t) = -c_1 t + c_2$. If $p_0 = 0$, then $u^* = \mathrm{sign}(p_2^*)$. Note that since t_f is free, we have $H \equiv 0$ along the optimal process, or

$$c x_2^*(t) + (-c_1 t + c_2) u^*(t) = 0.$$

In particular, at $t = t_f$, we get $-c_1 t_f + c_2 = 0$. Hence there is no switch. Then either $u^* \equiv 1$ or $u^* \equiv -1$. For the former case,

$$\begin{aligned} x_1(t) &= \xi_1 + \xi_2 t + \frac{1}{2} t^2 \\ x_2(t) &= \xi_2 + t \end{aligned}$$

for $t \in [0, t_f]$ and $x(t_f) = 0$. From this we can solve

$$\xi_1 - \frac{1}{2}\xi_2^2 = 0$$

and $t_f = -\xi_2$. Thus $\xi_2 < 0$, and $x_1 - \frac{1}{2}x_2^2 = 0$, with $x_2 < 0$ is the optimal trajectory. For the latter case $u^* \equiv -1$, we obtain the optimal trajectory in the same fashion: $x_1 + \frac{1}{2}x_2^2 = 0$, with $x_2 > 0$.

Now assume $p_0 = -1$. Then

$$u^*(t) = \begin{cases} -1, & p_2^*(t) \le -1 \\ 0, & -1 < p_2^*(t) \le 1 \\ 1, & p_2^*(t) > 1. \end{cases}$$

Therefore, the optimal control sequence can have the following patterns or its sub-sequence with the same order – meaning that there shouldn't be any sequence like $-1, 1$ since p_2^* is continuous:

$$\begin{aligned} &-1, 0, 1 \\ &1, 0, -1 \end{aligned}$$

Let's consider the first case, and the second case is similar. Denote the two switching times as $t_1 < t_2$. Then we can integrate the system on these intervals:

$$\begin{cases} x_1(t) = \xi_1 + \xi_2 t - \frac{1}{2}t^2, \\ x_2(t) = \xi_2 - t, \end{cases} \qquad 0 \le t < t_1$$
$$\begin{cases} x_1(t) = \xi_1 + \xi_2 t_1 - \frac{1}{2}t_1^2 + (\xi_2 - t_1)(t - t_1), \\ x_2(t) = \xi_2 - t_1, \end{cases} \qquad t_1 \le t < t_2$$
$$\begin{cases} x_1(t) = \xi_1 + \xi_2 t_1 - \frac{1}{2}t_1^2 + (\xi_2 - t_1)(t_2 - t_1) + (\xi_2 - t_1 - t_2)(t - t_2) + \frac{1}{2}(t^2 - t_2^2) \\ x_2(t) = \xi_2 - t_1 + t - t_2 \end{cases} \qquad t_2 \le t < t_f$$

At the switches, we have

$$p_2^*(t_1) = -1,\ p_2^*(t_2) = 1$$

or

$$-c_1 t_1 + c_2 = -1$$
$$-c_1 t_2 + c_2 = 1$$

Since t_f is free, the maximum principle tells us that

$$H = c_1 x_2^*(t_1) + (-c_1 t_1 + c_2)(-1) - 2 = 0$$

Putting $x_1(t_f) = 0,\ x_2(t_f) = 0$, then from the five equations we can solve c_1, c_2, t_1, t_2, t_f – quite cumbersome. The phase plot looks like in the following figure:

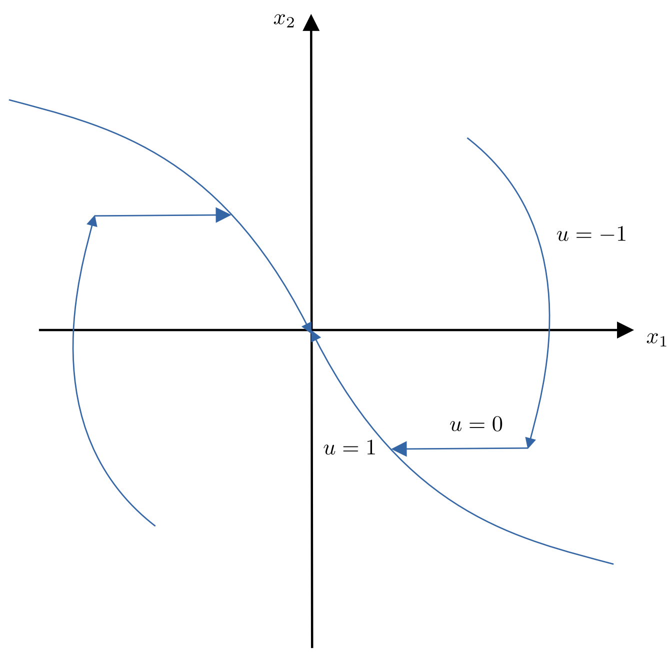

Figure 1: Minimum fuel-time control