Lecture 6 – Nonlinear controllability

What you will learn today (spoiler alert)

New mathematical concepts and language

- Manifolds, charts \((M, o(x))\)
- Vector fields \(\sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}\)
- Lie-derivative \(L_X(f) = \sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i}\)
- Lie-bracket \([f, g] = \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\)

Local Controllability:
- A nonlinear system is controllable if the linearized system is controllable.
- \(x = f(x) + g(x)u\) is “accessible” if
  \[\dim \{f, g, [f, g], [f, g], \ldots\} = n\]

Fundamental Parking Theorem

Important special affine case:
- \(\dot{x} = f(x) + g(x)u\)
- \(y = h(x)\)

Local Observability. Depends on \(x_0\) and \(u\).
- \(y_j = b_i(x)\)
- \(C = \text{span} L_{x_0} \cdots L_{x_0} b_j(x)\)
- \(dC = \text{span} \left\{ \frac{dH}{d\theta} \mid B \in C \right\}\)

The system is locally observable if \(\dim (dC) = n\)

Duality between observability and controllability

Basic Result: Linearization at \((x_0, u_0)\)

\[\dot{x} = f(x) + g(x)u, \quad x(0) = x_0\]

Theorem: Suppose \(f(x_0) + g(x_0)u_0 = 0\). If the linearization

\[\dot{z} = A_z + Bv\]

is controllable, then for all \(T > 0\), \(\epsilon > 0\) the set

\[N_T(x_0) = \{x(T) : |u - u_0| < \epsilon\}\]

contains a neighborhood of \(x_0\). (Proof: Nice exercise in using the inverse function theorem)
Manifolds

What are natural mathematical models for state spaces?

Piece together "bent" pieces of \( \mathbb{R}^n \). Same local properties as \( \mathbb{R}^n \). Different globally.

Example - Pendulum

\[ \ddot{\theta} = \sin(\theta) + u \]

Natural state space: \( \mathbb{R} \times S^1 = \text{cylinder} \)

\( S^1 \) = unit circle

Rigid Bodies

Natural State Space

\[ R = \{ r_1, r_2, r_3 \} \in SO(3) \]

\[ \omega \in \mathbb{R}^2 \] \( \omega \mapsto R = -RS(w) \)

\[ S(w) = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \]

Topological Space

A \( C^\infty \) manifold is a topological space \( M \) together with an atlas \( \{ U_\alpha, \phi_\alpha \} \) of pairwise \( C^\infty \)-compatible coordinate charts that cover \( M \).

**Definition of Manifold**

**Example: Cylinder**

\[ \psi \circ \varphi^{-1} \text{ smooth on } U \cap V \] \( z = \psi(\varphi^{-1}(x)) \) is given by \( (z_1, z_2) = (r_1, 4/r_2) \)

The cylinder is a smooth manifold.

**Compatible Coordinate Charts**

Compatible: \( \psi \circ \varphi^{-1}(x) \in C^\infty \)

\( f \) is called "smooth" if \( f(\psi(\varphi^{-1}(x))) \in C^\infty \), \( \forall \psi \)

Note: \( f \circ \varphi^{-1}(x) = f \circ \varphi^{-1} \circ \psi \circ \varphi^{-1} \in C^\infty \)

Independent on coordinate charts.

**Rolling Penny**

\[ \dot{x} = u_1 \cos(\theta) \]

\[ \dot{y} = u_1 \sin(\theta) \]

\[ \dot{\varphi} = u_1 \]

\[ \dot{\theta} = u_2 \]

The linearization is not controllable (check)

Can the penny be moved sideways in small time (keeping the head up)?

**Topology**

A topology on a set \( M \) is a collection \( T \) of subsets of \( M \).

\( O \) is called "open" if \( O \in T \).

The collection \( T \) must be such that

- \( \emptyset, M \in T \)
- \( O_1, O_2 \in T \iff O_1 \cap O_2 \in T \)
- \( \{ O_i \} \in T \iff \cup O_i \in T \)

**Rolling Penny**

Yes it can. But it is not obvious.

The penny has non-holonomic constraints \( \dot{z} = 0 \)

\[ \begin{bmatrix} \sin \theta & -\cos \theta & 0 & 0 \\ \cos \theta & \sin \theta & -1 & 0 \\ \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \\ \dot{\theta} \end{bmatrix} = 0 \]

Can not be written as holonomic constraints: \( h(z) = 0 \iff h_z = 0 \).
Differentials

A function \( f : A \to B \) is called differentiable at \( x \in A \) iff there is a continuous linear map \( DF_x : A \to B \) such that

\[
\| f(x+h) - f(x) - DF_x(h) \| \to 0, \quad h \to 0
\]

\( DF_x \) is called differential (Jacobian).

Definition Rank of \( f \) at \( x \) = rank(\( DF_x \)).

If \( f \) is smooth then

\[
\text{rank} (DF_0) = k \implies \text{rank} (DF_x) \geq k
\]

for all \( x \) close to \( x_0 \).

Proof: \( \text{det}(D_k(x)) = k \times k \) submatrix of \( DF_x \) with \( \text{det}(D_k(x)) \neq 0 \implies \text{det}(D_k(x)) \neq 0 \) for \( x \) close to \( x_0 \).

Global Differences to \( R^n \) - Example

Any smooth velocity field \( v \) on \( S^2 \) must have a point where \( v(x) = 0 \)

"You can't comb the hair of a tennis ball!"

Implicit Function Theorem

Example

\[
h(x,y) = 0
\]

\( \frac{\partial h}{\partial y} \) full rank \( \implies x = x(y) \) uniquely

\[
h(x,y) = x^2 + y^2 - 1, \quad \frac{\partial h}{\partial y} = 2x
\]

So \( x = x(y) \) uniquely except near \( (0, \pm 1) \).

In fact \( x = \sqrt{1-y^2}, \ x_0 > 0 \) and \( x = -\sqrt{1+y^2}, \ x_0 < 0 \).

Examples

Example

Double pendulum

Example Sphere \( S^2 \)

Differentials

Many manifolds are defined implicitly by equations systems

\[
\begin{align*}
f(x_1, \ldots, x_n) &= 0, \\
f(x_1, \ldots, x_n) &= 0
\end{align*}
\]

When does this describe a (smooth) \( n-k \)-dimensional manifold?

Inverse Function Theorem

Theorem Let \( X \) be open in \( U \) and \( f \in C^1(X,Y) \), \( f(x_0) = y_0 \). For existence of \( g \in C^1(Y,U) \) where \( Y \) is a neighborhood of \( y_0 \) so

a) \( f \circ g = \text{identity near } y_0 \)

b) \( g \circ f = \text{identity near } x_0 \)

c) a) and b)

it is necessary and sufficient that there is a linear map \( A \) such that respectively

a) \( f'(x_0)A = I_Y \)

b) \( Af'(x_0) = I_U \)

c) a) and b)

Condition c' implies that \( g \) is uniquely determined near \( y_0 \).
Different notation

\[ L_X(f) = X(f) \]

Lie derivative = fishermand derivative

Examples

\[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \neq \frac{\partial}{\partial y} \frac{\partial}{\partial x} \]

Globally, all derivatives are different

\[ \text{Example:} \quad \nabla f \neq \nabla g \]

\[ f_x \neq g_x \]

\[ f_y \neq g_y \]

\[ f_{xy} \neq g_{xy} \]

\[ f_{yx} \neq g_{yx} \]

\[ f_{xxy} \neq g_{xxy} \]

\[ f_{yyx} \neq g_{yyx} \]

\[ f_{xxx} \neq g_{xxx} \]

\[ f_{xyy} \neq g_{xyy} \]

\[ f_{yxy} \neq g_{yxy} \]

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\[ f_{yyyy} \neq g_{yyyy} \]
Assigns a tangent vector to each point: \( p \mapsto X_p \)

\[
X = \sum_{i=1}^{t} X_i(p) \frac{x_i}{x_i}
\]

\( X_i(p) \) smooth functions of \( p \).

Alternative notation: \( X \sim \{X_i(x_1, \ldots, x_n)\} \)

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**Example**

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

\[
\begin{align*}
g &= \frac{\partial h}{\partial x} - \frac{\partial f}{\partial u}(f + gu) = L_{f+g}h \\
y^{(k)} &= (L_{f+g})^kh
\end{align*}
\]

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**Main new object: Lie Bracket of vector fields**

Consider two vector fields \( \dot{x} = f(x) \) and \( \dot{y} = g(x) \).

Lie-bracket. New vector field

\[
[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g
\]

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**Example**

\[
\begin{align*}
x_1 &= u_1 \\
x_2 &= u_2 \\
x_3 &= x_1u_2 + x_2u_3
\end{align*}
\]

This means \( g_1 = \begin{pmatrix} 1 \\ 0 \\ x_2 \\ x_3 \end{pmatrix} \) and \( g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ x_1 \end{pmatrix} \).

\[
[g_1, g_2] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

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**Example**

Hence at \( x = 0 \) we have

\[
\begin{align*}
g_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
g_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
g_1 \cdot g_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{align*}
\]

With the minus-sign the three vector fields span \( \mathbb{R}^3 \), and we have controllability.

With the plus-sign the system is not controllable, in fact it can be seen that \( x_1' + x_2' - 2x_3 \) is an invariant.

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**Example**

\[
\begin{align*}
X'(p) &= \text{solution to } \dot{x} = X(x), x(0) = p \\
X' \text{ is smooth. } X'^0 &= \text{id} \\
L_X(g) &= X(g) = \sum_{i=1}^{t} X_i \frac{\partial g}{\partial x_i} = \lim_{h \to 0} \left( X^h(p) g - g(p) \right) \\
L_{aX+bY} &= aL_X + bL_Y, \ a, b \in \mathbb{R}
\end{align*}
\]

\[
\dot{z} = f(x, u)
\]

\[
f : M \times U \mapsto TM
\]

---

**Why is it interesting?**

Roughly we have:

If the Liebracket "tree" has full rank, then the system is "controllable".
Lie-Brackets

\[ [X,Y]_b(f) = X_b(Y(f)) - Y_b(X(f)) \]
\[ X \sim \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}, \quad Y \sim \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \]
\[ [X,Y] = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \]

Some Lie-Bracket Formulas

\[ [fX,gY] = fg[X,Y] + fX(gY) - gY(fX) \]
\[ [X,Y] = -[Y,X] \]
\[ X_1, X_2, X_3 \mid \sum X_i [X_2, X_3] \mid [X_2, [X_3, X_1]] = 0 \]
\[ L_X Y = [X,Y] = \lim_{h \to 0} \frac{1}{h} X^h Y - Y \]
\[ X^h Y = \sum_{n=0}^{\infty} \frac{h^n}{n!} Y = Y + h [X,Y] + \frac{h^2}{2!} [X,[X,Y]] + \ldots \]
related to
\[ e^{X^hY} = e^{\Delta C}; \quad C = A + B \]

Another example

\[ X = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \]
\[ Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ [X,Y] = \begin{pmatrix} 1 & 0 \\ 0 & -\sin \phi \end{pmatrix} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \sim \begin{pmatrix} \cos \phi - \sin \phi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \]

Why are Lie-brackets so fundamental?

\[ \dot{x} = g_1 u_1 + g_2 u_2 \]
\[ (u_1(t), u_2(t)) = \begin{cases} (1,0) & t \in [0,b) \\ (1,0) & t \in [2b,3b) \\ (0,-1) & t \in [3b,4b) \end{cases} \]
\[ x(4b) = x_0 + h^2 [g_1, g_2] + O(h^3) \]

Proof sketch
\[ \left( 1 + \frac{hf}{1!} + o\left(\frac{hf}{1!}\right) \right)^n \to e^{hf} \]

Vector Fields, Summary

A vector field \( X \) is associated with:

a) A system of differential equations
\[ \frac{dx}{dt} = X(x) \]
b) A flow \( \Phi^t : M \to M, t \in [0,t_1], \) where \( \sigma(t) = \Phi^t(x) \) is the solution to
\[ \frac{d\sigma}{dt} = X(\sigma), \quad \sigma(0) = x \]
c) A directional derivative
\[ X_{\cdot f} = \frac{d}{dt}(\Phi^t(x))|_{t=0} \]
d) A "derivation" of the algebra \( C^\infty(M) \)
e) A partial differential operator
\[ X = \sum X_i \frac{\partial}{\partial x_i} \]

Parking cont’d

\[ \text{Steer} := \frac{\partial}{\partial \theta} \]
\[ \text{Drive} := \cos(\phi + \theta) \frac{\partial}{\partial \phi} + \sin(\phi + \theta) \frac{\partial}{\partial \psi} \]
\[ \psi = k \sin(\theta) \]
\[ \text{Steer} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{Drive} = \begin{pmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ \sin(\theta) \end{pmatrix} \]

[Steer, Drive] = \begin{pmatrix} -\sin(\phi + \theta) \\ \cos(\phi + \theta) \\ 0 \end{pmatrix} \Rightarrow \text{Wriggle}
An easy calculation (exercise) shows that
\[
[Wriggle, Drive] = \begin{bmatrix}
-sin(\phi + \theta) \\
-\cos(\phi + \theta) \\
0 \\
0
\end{bmatrix} =: \text{Slide}
\]

For \( \theta = 0 \) this takes you sideways:

\[
\text{Slide}^t(x, y, \phi, 0) = (x - t \sin(\phi), x + t \cos(\phi), 0, 0)
\]

\[f, g\] = \begin{bmatrix}
A, & B \\
B, & 0
\end{bmatrix} = 0 \
\neq AB
\]
\[g, [f, g] = 0
\]
\[f, [f, g] = A^2 B
\]

**Controllability Theorems**

**Accessibility Theorem**

\[
C = \text{smallest Lie subalg. containing } \{f, g_1, \ldots, g_m\}
\]

**Theorem** If for all \( x_0 \) the Lie-bracket tree contains \( n \) linearly independent elements, then the system is has the accessibility property

\[
\dim C = n \implies \text{can reach open set}
\]

If \( f = 0 \), or more generally \( f(x, u) \) is "symmetric", see Glad) then the system is controllable: \( A(x_0) = \mathbb{R}^n \)

**Fundamental Parking Theorem**

You can get out of any parking lot that is larger than the car. Use the following control: Wriggle, Drive, –Wriggle (this requires a cool head), –Drive (repeat).

Proof: Trotters Product Formula

\[
\dot{x} = Ax + Bu - f(x) + g(x)u
\]

For a precis formulation, and more about "controllability" vs "accessability" see

T. Glad, Nonlinear Control Theory, Chapter 8, pp 73-81