Nonlinear Control Theory 2017

- L1 Nonlinear phenomena and Lyapunov theory
- L2 Absolute stability theory, dissipativity and IQCs
- L3 Density functions and computational methods
- L4 Piecewise linear systems, jump linear systems
- L5 Relaxed dynamic programming and Q-learning
- L6 Controllability and Lie brackets
- L7 Synthesis: Exact linearization, backstepping, forwarding

Exercise sessions:
Solve 50% of problems in advance, or make hand-in later.

Mini-project:
(4-5 days) Study and present topic related to your research.

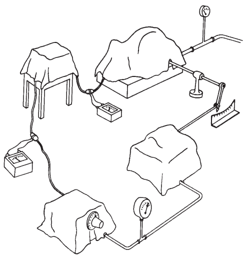
Written take-home exam.

L2: Absolute stability, dissipativity and IQCs

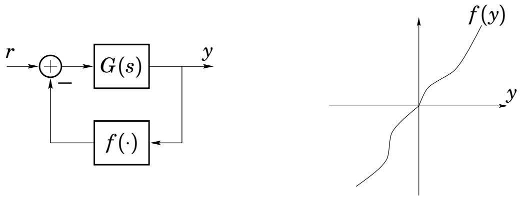
- Absolute Stability Theory
- Dissipativity theory
- Integral Quadratic Constraints
- Examples
- Dissipativity from IQCs
- Toolbox

Literature.
On IQCs: Megretski/Rantzer, IEEE TAC 42:6 (1997)
On dissipativity: Willems, Archive Rational Mech.Anal. 45:5 (1972)
See also course web page.

Stability and Performance of Complex Systems



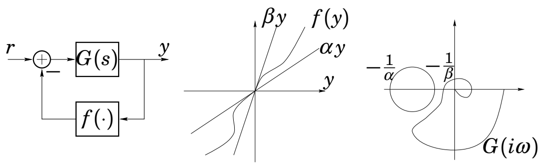
Absolute Stability Theory



For what $G(s)$ and $f(\cdot)$ is the closed-loop system stable?

- Lur'e and Postnikov's problem (1944)
- Aizerman's conjecture (1949) (False!)
- Kalman's conjecture (1957) (False!)
- Solution by Popov (1960) (Led to the Circle Criterion)

The Circle Criterion



Theorem.
Let $y = G(s)u$ and $u = -f(y) + r$. Assume $G(s)$ is stable and $0 < \alpha \le \frac{f(y)}{y} \le \beta < \infty$. If $G(i\omega)$ does not encircle the disc defined by $-1/\alpha$ and $-1/\beta$, then the closed-loop is BIBO stable from r to y.

L2: Absolute stability, dissipativity and IQCs

- Absolute Stability Theory
- **Dissipativity theory**
- Integral Quadratic Constraints
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Dissipativity

The nonlinear system

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)). \end{cases}$$

is said to be *dissipative* with respect to the *supply rate* $r(u,y)$ if there exists a *storage function* $S(x) \ge 0$ such that

$$S(x(t_0)) + \int_{t_0}^{t_1} r(u(t), y(t))dt \quad \ge \quad S(x(t_1))$$

Interconnection

Suppose

$$\dot{x}_1 = f_1(x_1, u_1) \qquad\qquad \dot{x}_2 = f_2(x_2, u_2)$$

are dissipative with supply rates $r_1(u_1, x_1)$ and $r_2(u_2, x_2)$ and storage functions $S(x_1)$, $S(x_2)$. Then

$$\begin{cases} \dot{x}_1 = f_1(x_1, h_2(x_2)) \\ \dot{x}_2 = f_2(x_2, h_1(x_1)) \end{cases}$$

is dissipative with respect to the supply rate

$$\tau_1 r_1(h_2(x_2), x_1) + \tau_2 r_2(h_1(x_1), x_2) \qquad \tau_1, \tau_2 \ge 0$$

and storage function

$$\tau_1 S_1(x_1) + \tau_2 S_2(x_2)$$

Storage and Lyapunov functions

For a system without input, suppose that

$$r(y) \le -k|x|^c$$

for some $k > 0$. Then the dissipation inequality implies

$$S(x(t_0)) - \int_{t_0}^{t_1} k|x(t)|^c dt \ge S(x(t_1))$$

which is an integrated form of the Lyapunov inequality

$$\frac{d}{dt} S(x(t)) \le -k|x|^c$$

Example—Capacitor

A capacitor

$$i = C\frac{du}{dt}$$

is dissipative with respect to the supply rate $r(t) = i(t)u(t)$.

A storage function is

$$S(u) = \frac{Cu^2}{2}$$

In fact

$$\frac{Cu(t_0)^2}{2} + \int_{t_0}^{t_1} i(t)u(t)dt = \frac{Cu(t_1)^2}{2}$$

Mini-problem:
Give a disspation inequality for an inductor $v = L\frac{di}{dt}$.
What about an RLC circuit?

Memoryless Nonlinearity

The memoryless nonlinearity $y = \phi(u)$ with sector condition

$$\alpha \le \phi(v)/v \le \beta$$

is dissipative with respect to the quadratic supply rate

$$r(u,y) \quad = \quad -[y - \alpha u][y - \beta u]$$

with storage function

$$S \quad \equiv \quad 0$$

The Kalman-Yakubovich-Popov lemma

Given A, B and $M = M^T$, with $i\omega I - A$ nonsingular for $\omega \in \mathbf{R}$,the following statements are equivalent.

(*i*) For all $\omega \in [0, \infty]$ it holds that

$$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0$$

(*ii*) There exists a symmetric matrix $P \in \mathbf{R}^{n\times n}$ such that

$$M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \prec 0$$

Note that if $P \succeq 0$, then (ii) means that the linear system $\dot{x} = Ax + Bu$ is disspative with respect to the storage function x^TPx and supply rate $-\begin{bmatrix} x^T & u^T \end{bmatrix} M \begin{bmatrix} x^T & u^T \end{bmatrix}^T$.

The Circle Criterion Revisited

Theorem.
Let $y = G(s)u$ and $u = -f(y) + r$. Assume $G(s)$ is stable and $0 < \alpha \le \frac{f(y)}{y} \le \beta < \infty$. If $G(i\omega)$ does not encircle the disc defined by $-1/\alpha$ and $-1/\beta$, the closed-loop is BIBO stable.

Proof using dissipativity argument.
The frequency condition on $G(s) = C(sI - A)^{-1}B$ means (by the KYP lemma) that $\dot{x} = Ax + Bu$, $y = Cx$ is dissipative with storage function x^TPx and supply rate $[y - \alpha u][y - \beta u] - \epsilon|x|^2$.

At the same time, the nonlinearíty $y = \phi(u)$ is dissipative with storage function zero and supply rate $-[y - \alpha u][y - \beta u]$.

Adding the the two inequalities shows that the interconnected system $\dot{x} = A - Bf(Cx)$ satisfies $\frac{d}{dt}x^TPx \le -\epsilon|x|^2$.

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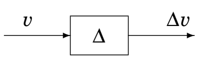
Integral Quadratic Constraint



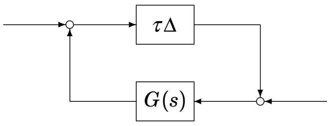
The (possibly nonlinear) operator Δ on $\mathbf{L}_2^m[0,\infty)$ is said to *satisfy the IQC defined by* Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ (\widehat{\Delta v})(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ (\widehat{\Delta v})(i\omega) \end{bmatrix} d\omega \ge 0$$

for all $v \in \mathbf{L}_2[0,\infty)$.

Δ structure	$\Pi(i\omega)$	Condition
Δ passive	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	
$\lVert\Delta(i\omega)\rVert \le 1$	$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$	$x(i\omega) \ge 0$
$\delta \in [-1,1]$	$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$	$X = X^* \ge 0$ $Y = -Y^*$
$\delta(t) \in [-1,1]$	$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$	
$\Delta(s) = e^{-\theta s} - 1$	$\begin{bmatrix} x(i\omega)\rho(\omega)^2 & 0 \\ 0 & -x(i\omega) \end{bmatrix}$	$\rho(\omega) =$ $2\max_{\lvert\theta\rvert \le \theta_0} \sin(\theta\omega/2)$

IQC Stability Theorem



Let $G(s)$ be stable and proper and let Δ be causal.

For all $\tau \in [0,1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

then the feedback system is input/output stable.

Proof steps

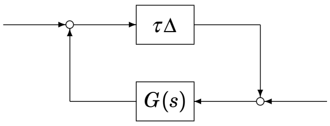
Step 1 Verify existence of $C > 0$ such that

$$|v| \le C|v - \tau G\Delta v| \qquad \forall v \in \mathbf{L}_2^l[0,\infty), \tau \in [0,1]$$

Step 2 Show that if $(I - \tau G\Delta)^{-1}$ is bounded for some $\tau \in [0,1]$, then $(I - \nu G\Delta)^{-1}$ is bounded for all ν in an interval around τ. The size of the interval depends on C, but not on τ.

Step 3 Starting from $\tau = 0$, prove by induction boundedness for all $\tau \in [0,1]$.

Stability Verification Using IQCs



Let $G(s)$ be stable and proper and let Δ be causal.

Collect (for example from comuputer library) a set of weights $\Pi_1(i\omega),\dots,\Pi_N(i\omega)$ corresponding to IQCs satisfied by $\tau\Delta$.

Use convex optimization to find $\tau_1,\dots,\tau_N \ge 0$ such that

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \sum_{k=1}^{N} \tau_k \Pi_k(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty].$$

The feedback system is input/output stable if solution is found.

Miniproblem: How do you state the optimization problem?

Performance Verification using "S-procedure"

The inequality

$$\sigma_0(h) \le 0$$

follows from the inequalities

$$\sigma_1(h) \ge 0, \dots, \sigma_n(h) \ge 0$$

if there exist $\tau_1,\dots,\tau_n \ge 0$ such that

$$\sigma_0(h) + \sum_k \tau_k \sigma_k(h) \le 0 \qquad \forall h$$

S-procedure losslessness by Megretsky/Treil

Let $\sigma_0, \sigma_1, \dots, \sigma_n : \mathbf{L}_2^m \to \mathbf{R}$ be continuous time-invariant quadratic forms and let $L \subset \mathbf{L}_2^m$ be a time-invariant subspace. Suppose that there exists $f_* \in L$ such that $\sigma_k(f_*) > 0$ for $k = 1,\dots,m$. Then the following statements are equivalent

(i) $\sigma_0(f) \le 0$ for all f such that $\sigma_1(f) > 0, \dots, \sigma_n(f) > 0$.

(ii) There exist $\tau_1,\dots,\tau_n \ge 0$ such that

$$\sigma_0(f) + \sum_k \tau_k \sigma_k(f) \le 0 \qquad \forall f \in L.$$

Mini-problem.
1. Is $(i) \Leftrightarrow (ii)$ when $\sigma_0,\dots,\sigma_n$ are linear forms on $\mathbf{R}^m$?
2. Is $(i) \Leftrightarrow (ii)$ when $\sigma_0,\dots,\sigma_n$ are quadratic forms on $\mathbf{R}^m$?

Proof of S-procedure losslessness

Define

$$K = \{(\sigma_0(f), \sigma_1(f), \dots, \sigma_n(f))\}$$
$$K_0 = \{(x_0, x_1, \dots, x_n) : x_0 > 0, x_1 > 0, \dots, x_n > 0\}$$

The statement ı is that $K \cap K_0 = \emptyset$. For $f \in L$, define $f^\tau \in L$ by $f^\tau(s) = f(s - \tau)$ for $s > \tau$. The closure $\bar{K}$ of the set K is convex, because

$$\lim_{\tau\to\infty} \sigma\left(\frac{g + f^\tau}{\sqrt{2}}\right) = \lim_{\tau\to\infty} \frac{1}{2}(\sigma(g) + \sigma(f^\tau)) = \frac{1}{2}(\sigma(g) + \sigma(f))$$

Hence ı implies existence of a hyperplane in $\mathbf{R}^{n+1}$ separating K_0 and $\bar{K}$. This implies (ii) .

The opposite implication is trivial.

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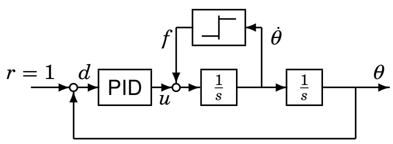
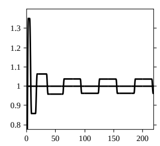
Example — Oscillations due to Stiction



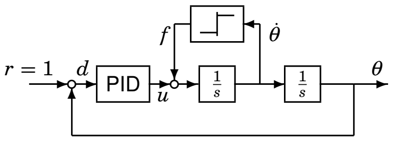


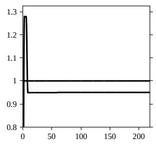
$$d(t) = \theta(t) - 1$$

$$u(t) = -K\left(T_d \dot{d}(t) + d(t) + \frac{1}{T_i}\int_0^t d(\tau)d\tau\right)$$

$$\ddot\theta(t) = u(t) - \text{stic}(\dot\theta(t))$$

Integrator Leakage Removes Oscillations



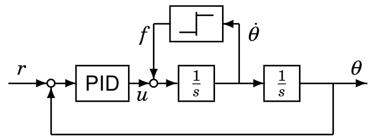


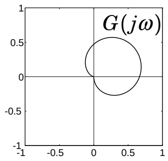
Controller

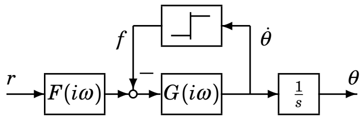
$$u(t) = -K\left(T_d\dot{d}(t) + d(t) + \frac{1}{T_i}\int_0^t e^{\epsilon(\tau-t)}d(\tau)d\tau\right)$$

We will use *integral quadratic constraints* to quantify the leakage level ϵ needed to remove oscillations.

Passivity not enough for stiction analysis







$$G(s) = \frac{s(s+\epsilon)}{(sT+1)(s^2+2\zeta\omega_0 s+\omega_0^2)}$$

Zames/Falb's IQC for Saturations

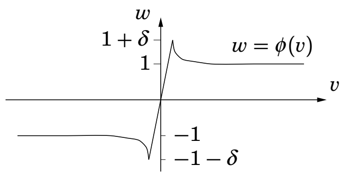
$$\begin{cases} f(t) = -1 & \text{if } v(t) < -1 \\ f(t) = v(t) & \text{if } v(t) \in [-1,1] \\ f(t) = 1 & \text{if } v(t) > 1 \end{cases}$$

Zames/Falb's property

$$0 \le \int_0^\infty [v(t) - f(t)][f(t) + (h * f)(t)]dt, \qquad \int_{-\infty}^\infty |h(t)|dt \le 1$$

$$0 \le \int_{-\infty}^\infty \begin{bmatrix}\hat{v}\\ \hat{f}\end{bmatrix}^* \begin{bmatrix} 0 & 1+H(-i\omega) \\ 1+H(i\omega) & -2(1+\operatorname{Re} H(i\omega))\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{f}\end{bmatrix} d\omega$$

IQC's for Stiction



$$\begin{cases} \phi(v) \in [-1-\delta, -1] & \text{if } v < 0 \\ \phi(v) \in [-1-\delta, 1+\delta] & \text{if } v = 0 \\ \phi(v) \in [1, 1+\delta] & \text{if } v > 0 \end{cases}$$

Integral Quadratic Constraints:

$$\int_0^\infty v(t)\left[(1+\delta)\phi(t) + (h * \phi)(t)\right]dt \;\ge\; 0, \qquad \int_{-\infty}^\infty |h(t)|dt \le 1$$

Stiction Stability Theorem

For $G(s)$ stable and proper and ϕ satisfying the conditions on the previous slide, consider the interconnection

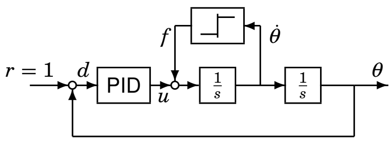
$$\begin{cases} v = Gw + f \\ w = \phi(v) + e \end{cases}$$

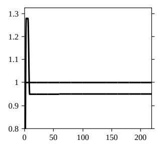
Assume well-posedness all $\tau \in [0,1]$. If there exists $H \in \mathbf{RL}_\infty$ with $\|H\|_{\mathbf{L}_1} \le 1$ and

$$\operatorname{Re}\left[G(i\omega)\left(1+\delta+H(i\omega)\right)\right] > 0, \quad \omega \in [0,\infty]$$

then the interconnection is stable.

Integrator Leakage $\epsilon > \delta/T$ Removes Oscillations





Controller

$$u(t) = -K\left(T_d\dot{d}(t) + d(t) + \frac{1}{T_i}\int_0^t e^{\epsilon(\tau-t)}d(\tau)d\tau\right)$$

Characteristic Polynomial

$$(Ts+1)(s^2+2\zeta\omega_0 s+\omega_0^2)$$

Leakage $\epsilon T > \delta$ Removes Stiction Oscillations

Let $1 > \epsilon T > \delta$ and $H(i\omega) = \frac{(1+\delta)(\epsilon T+1)}{-i\omega T+1}$. Then

$$\|H\|_{\mathbf{L}_1} = (1+\delta)(1-\epsilon T) < 1-\delta^2 < 1$$

$$\begin{aligned}\operatorname{Re}[G(1+\delta+H)] &= (1+\delta)\operatorname{Re}\left[G\left(1+\frac{\epsilon T-1}{-i\omega T+1}\right)\right] \\ &= \frac{T(1+\delta)(\omega^2+\epsilon^2)}{\omega^2T^2+1}\operatorname{Re}\frac{G(i\omega T+1)}{i\omega+\epsilon} \\ &= \frac{T(1+\delta)(\omega^2+\epsilon^2)}{\omega^2T^2+1}\operatorname{Re}\frac{i\omega}{-\omega^2+i2\zeta\omega_0\omega+\omega_0} > 0\end{aligned}$$

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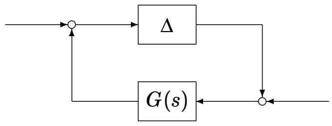
Recall IQC Stability Theorem



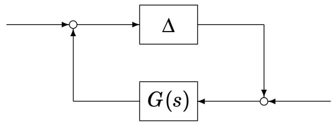
Let $G(s)$ be stable and proper and Δ causal. For all $\tau \in [0,1]$, suppose the loop is well posed and

$$0 \le \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(i\omega) \\ \widehat{(\tau\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \hat{v}(i\omega) \\ \widehat{(\tau\Delta v)}(i\omega) \end{bmatrix} d\omega \qquad \forall v$$

$$0 \succ \begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} \qquad \forall \omega$$

then the feedback system is input/output stable.

Case 1: Constant Weight

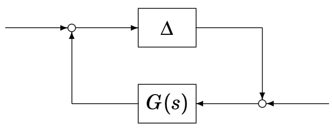


Forget τ for a moment. The inequalities can be written

$$0 \le \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* M \begin{bmatrix} \hat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega$$

$$0 \succ \begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* M \begin{bmatrix} G(i\omega) \\ I \end{bmatrix}$$

Case 1: Constant Weight

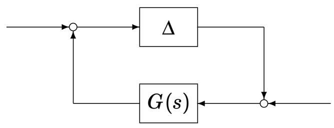


Rewrite using Parseval's formula and the KYP Lemma:

$$0 \le \int_0^{\infty} \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}^* M \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} dt$$

$$0 \succ \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^T M \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A^TP+PA & PB \\ B^TP & 0 \end{bmatrix}$$

Case 1: Constant Weight

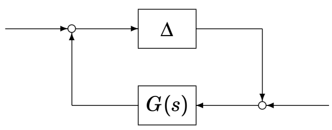


Multiply the second inequality with (x,w) from right and left

$$0 \le \int_0^{\infty} \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}^T M \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} dt$$

$$0 \succ \begin{bmatrix} v \\ w \end{bmatrix}^T M \begin{bmatrix} v \\ w \end{bmatrix} + \frac{d}{dt} x^T P x$$

Case 1: Constant Weight



Assume the IQC is "hard" (holds on finite intervals) and $P \succ 0$:

$$0 \le \int_0^{T} \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}^* M \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} dt$$

$$0 > \int_0^{T} \begin{bmatrix} v \\ w \end{bmatrix}^T M \begin{bmatrix} v \\ w \end{bmatrix} dt + x(T)^T P x(T) - x(0)^T P x(0)$$

The second inequality proves dissipativity of the linear part. Adding the first inequality shows that P is a Lypaunov function.

Mini-problem:
The assumptions hold for certain classes of M. Which ones?

Case 2: Frequency Dependent Weight

In the general case $\Pi(i\omega)$ can be factorized as

$$\Pi(i\omega) = \Psi(i\omega)^* M \Psi(i\omega) \qquad (\star)$$

and the KYP Lemma is a applied to an extended state realization involving both the states of G and the states of Ψ. Again, assuming that

$$\Pi(i\omega) = \begin{bmatrix} \Pi_{11}(i\omega) & \Pi_{12}(i\omega) \\ \Pi_{21}(i\omega) & \Pi_{22}(i\omega) \end{bmatrix}$$

with $\Pi_{11}(i\omega) \succ 0$ and $\Pi_{22}(i\omega) \prec 0$ it is possible to prove[1] that the factorization $(\star)$ can be made to get a valid hard IQC and $P \succ 0$. Hence the system is dissipative with a storage function that is quadratic in the extended state.

[1] Seiler IEEE TAC 60:6 (2015)

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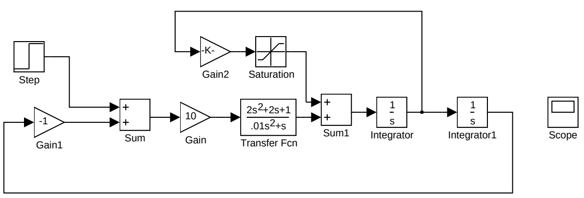
- Examples
- Dissipativity from IQCs
- **Toolbox**

A servo with friction

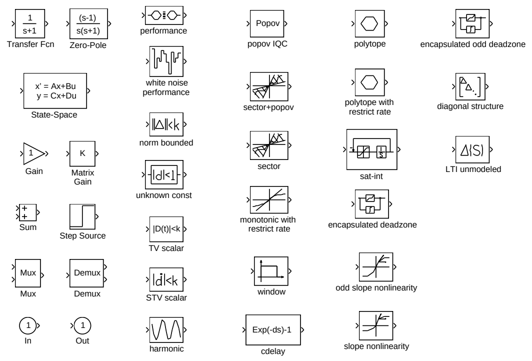


Simulations show stability.

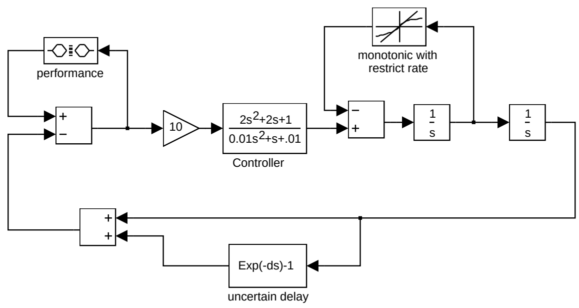
The circle criterion can *prove* stability.

But what if the feedback controller induces time delays?

A library of analysis objects

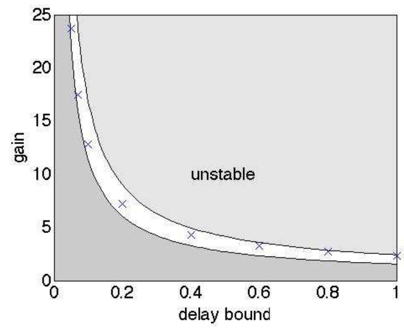


The IQC toolbox



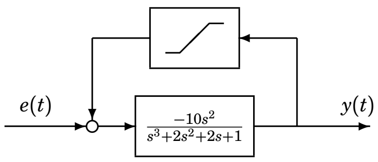

```
>> iqc_gui('fricSYSTEM')

extracting information from fricSYSTEM ...

    scalar inputs:  5
    states:         10
    simple q-forms: 7

Solving with 62 decision variables ...

  ans =     4.7139
```

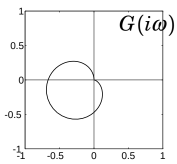
Verification by IQCs



IQCs prove stability below the lower line.

An analysis model in text format






```
G = tf([10 0 0],[1 2 2 1]);
e = signal;
w = signal;
y = -G*(e+w);
w==iqc_monotonic(y);

iqc_gain_tbx(e,y)
```

L2: Absolute stability, dissipativity and IQCs

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