Session 3

Density functions and sum-of-squares methods

Reading assignment

Check the main results and examples of these papers.

- Rantzer, Systems & Control Letters, 42:3 (2001).
- Prajna/Parrilo/Rantzer, TAC 49:2 (2004).
- SOSTOOLS and its Control Applications, Prajna/P/S/P (2005)

Exercise 3.1
a. Draw a phase plot for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\epsilon x_1 + x_1^2 - x_2^2 \\ -\epsilon x_2 + 2x_1 x_2 \end{bmatrix}$$

for $\epsilon = 1$.
b. Prove for $\epsilon > 0$ that $\lim_{t\to\infty} x(t) = 0$ for almost all initial states.
c. What happens for $\epsilon = 0$?

Exercise 3.2
In the lecture slide "Example — Patching nonlinear controllers" we claimed that density functions enable construction of an "almost globally stabilizing" feedback law $u(x)$ that acts as $u_1(x) = -3x_1 - 6x_2$ for small x and $u_2(x) = x_1 - 2x_2$ for large x. Complete the constrution and find the set of initial states for which $\lim_{t\to\infty} x(t) \neq 0$.

Exercise 3.3
a. Verify that the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -6x_1 x_2^2 - x_1^2 x_2 + 2x_2^3 \\ x_2 u \end{bmatrix}$$

is not asymptotically stabilizable.
b. Find a control law such that $\lim_{t\to\infty} x(t) = 0$ for almost all initial states in the closed loop system. Draw a phase plot.

Exercise 3.4
Error dynamics for an attitude observer in two dimensions can be represented by the same equation as the one used in the lecture

$$\dot{R}(t) = R(t)[R(t)^T - R(t)] + R(t)E(t),$$

but now the orthogonal matrix $R(t)$ is only 2×2.
a. Give $\epsilon, \delta > 0$ such that $\limsup_{t\to\infty} \|R(t) - I\| < \delta$ for almost all initial conditions when $E(t) = -E(t)^T$ is a constant matrix with norm smaller than ϵ.
b. What if $E(t)$ is not constant?