Control System Design - LQG

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Lecture - LQG Design

- Introduction
- The H_2-norm
- Formula for the optimal LQG controller
- Software, Examples
- Properties of the LQ and LQG controller
- Design tricks, how to tune the knobs
- What do the “technical conditions” mean?
- How to get integral action etc
- Loop Transfer Recovery (LTR)
- More Examples

For theory and more information, see PhD course on LQG

Reading tips: Ch 5 in Maciejowski

Linear Quadratic Gaussian Design

Process model

$$\begin{aligned} \dot{x} &= Ax + Bu + v \\ y &= Cx + Du + w \end{aligned}$$

where v,w is white gaussian noise with mean zero

$$E \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \begin{pmatrix} v(\tau) \\ w(\tau) \end{pmatrix}^T = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \delta(t-\tau)$$

Optimization criterion

$$\min E \int_0^T \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt$$

Short on Stochastics

$$\begin{aligned} \dot{x} &= Ax + v \\ y &= Cx + e \end{aligned}$$

v white noise, $\quad \mathsf{E}v(t)v^T(t-\tau) = R_1\delta(\tau)$

e white noise, $\quad \mathsf{E}e(t)e^T(t-\tau) = R_2\delta(\tau)$

State covariance

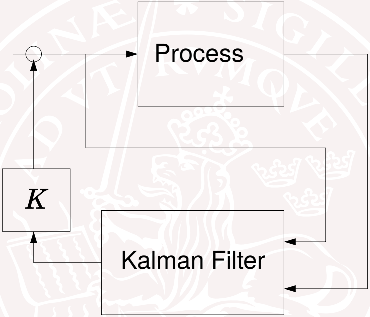
$\mathsf{E}x(t)x^T(t) = R(t), \quad \dot{R} = AR + RA^T + R_1$

Kalman filter, $\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x})$

$\tilde{x} = x - \hat{x}, \quad \mathsf{E}\tilde{x}(t)\tilde{x}^T(t) = P(t)$

$\dot{P} = AP + PA^T + R_1 - PC^TR_2^{-1}CP, \quad L = PC^TR_2^{-1}$

Separation Principle



Nice structure of the optimal controller: $u = -K\widehat{x}$

Linear feedback combined with state estimation

Certainty equivalence principle

Linear Quadratic Gaussian Design - History

In the late 50s and early 60s computers where starting to be used to find "optimal" controllers.

Classical Reference: Newton, Gould, Kaiser (1957)

Wiener-Kolmogorov

Kalman-Bucy

Bellman, Wonham, Willems, Andersson, Åström, Kucera and many others

Why so popular?

Gives “optimal” controller

Automized design method. Works for MIMO.

Nice formulas for the optimal controller, reasonable computational effort

Gives absolute scale of merit - know limits of performance

Used for space program, aircraft design - Good models often available

LQ control give good robustness margins (with $Q_12 = 0$)

- $[1/2, \infty]$-gain margin
- 60 degree phase margin

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LQG is optimizing the H_2 norm

Consider the system

$$\begin{aligned} Y &= G(s)U \\ y &= g \star u \\ \dot{x} &= Ax + Bu \\ y &= Cx + (Du) \end{aligned}$$

The L_2-norm (LQG-norm) is defined as

$$\begin{aligned} \|G\|_2^2 &= \sum_i \sum_j \int_{-\infty}^{\infty} |G_{ij}(j\omega)|^2 d\omega/2\pi = \\ &= \int_{-\infty}^{\infty} \text{trace}\,\{G^*(j\omega)G(j\omega)\} d\omega/2\pi \end{aligned}$$

H_2: Equals L_2-norm if G asymptotically stable, equals ∞ otherwise

H_2-norm as noise power gain

u: stationary white noise, mean zero

$$E(u(\tau_1)u(\tau_2)^T) = \delta(\tau_1 - \tau_2)I$$
$$S_u(\omega) = 1, \forall\omega$$

then $\mathrm{Pow}(y) = E(y^T y) = \|G\|_2^2$.

”Amplification of noise power”

Proof

$$
\begin{aligned}
E(\text{tr } yy^T) &= \text{tr} \int S_y(\omega) d\omega/2\pi = \\
&= \int \text{tr } G^*(j\omega) S_u(\omega) G(j\omega) d\omega/2\pi \\
&= \|G\|_2^2
\end{aligned}
$$

Another interpretation of the H_2-norm

By Parseval's formula we have

$$\|G\|_2^2 \;=\; \sum_i \sum_j \int_{-\infty}^{\infty} |g_{ji}(t)|^2 dt$$

Hence the H_2-norm can also be interpreted as "energy in impulse responses":

$$\|G\|_2^2 = \sum_{i=1}^{m} \int_0^{\infty} |g_{:,i}|^2(t)dt$$

How to compute the H_2 norm

1) `norm(sys)` in Matlab

2) If $G(s) = C(sI - A)^{-1}B$ then

$$\|G\|_2^2 = \text{trace}\,(CPC^T) = \text{trace}\,(B^T S B)$$

where P is the unique solution to the Lyapunov equation

$$AP + PA^T + BB^T = 0$$

and S solves

$$SA + A^T S + C^T C = 0$$

3) Residue calculus

$$\|G\|_2^2 = \sum_{i,j} \frac{1}{2\pi i} \oint G_{ij}(-s)^T G_{ij}(s) ds$$

4) Recursive formulas ala Åström-Jury-Schur

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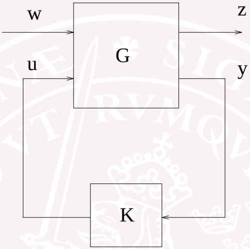
A unified framework



$$u \quad = \quad \text{Control inputs}$$

$$y \quad = \quad \text{Measured outputs}$$

$$w \quad = \quad \text{Exogenous inputs} = \begin{cases} \text{Fixed commands} \\ \text{Unknown commands} \\ \text{Disturbances} \\ \text{Noise} \ldots \end{cases}$$

$$z \quad = \quad \text{Regulated outputs} = \begin{cases} \text{Tracking errors} \\ \text{Control inputs} \\ \text{Measured outputs} \\ \text{States} \ldots \end{cases}$$

The H_2 Problem

Closed Loop

$$\begin{aligned} u &= K(s)y \\ z &= G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}w = T_{zw}w \end{aligned}$$

The H_2 problem:

Find $K(s)$ such that the closed loop is stable and

$$\min_{K(s)} \|T_{zw}\|_2$$

is obtained.

The Optimal Controller

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x + \phantom{D_{21}w +} D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{aligned}
$$

Under some technical conditions the optimal controller is

$$
\begin{aligned}
u &= -K\widehat{x} \\
\dot{\widehat{x}} &= A\widehat{x} + B_2 u + L(y - C_2\widehat{x} - D_{22}u) \\
K &= (D_{12}^T D_{12})^{-1}(D_{12}^T C_1 + B_2^T S) \\
L &= (B_1 D_{21}^T + P C_2^T)(D_{21} D_{21}^T)^{-1}
\end{aligned}
$$

where $P \geq 0$ and $S \geq 0$ satisfy

$$
\begin{aligned}
0 &= SA + A^T S + C_1^T C_1 - K^T D_{12}^T D_{12} K \\
0 &= AP + PA^T + B_1 B_1^T - L D_{21} D_{21}^T L^T
\end{aligned}
$$

$A - B_2 K, \qquad A - LC_2$ stable

The Optimal Controller

$$u = -K(sI - A + B_2K + LC_2 - LD_{22}K)^{-1}Ly$$

Controller has same order as process

(How to introduce reference signals later)

“Technical Conditions”

1) $[A, B_2]$ stabilizable

2) $[C_2, A]$ detectable

3) “No zeros on imaginary axis” $u \to z$

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m \qquad \forall \omega$$

and D_{12} has full column rank (no free control)

4) “No zeros on imaginary axis” $w \to y$

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p \qquad \forall \omega$$

and D_{21} has full row rank (no noise-free measurements)

Alternative formulation

Weight matrices

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} = \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix} \begin{pmatrix} C_1 & D_{12} \end{pmatrix}$$

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} = \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} \begin{pmatrix} B_1^T & D_{21}^T \end{pmatrix}$$

If $Q_{12} = 0$ then the notation Q_1 and Q_2 is sometimes used instead, similar for R

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Software

Matlab - Control system toolbox

```
lqr, dlqr - Linear-quadratic (LQ) state-feedback regulator
lqry      - LQ regulator with output weighting
lqrd      - Discrete LQ regulator for continuous plant
kalman    - Kalman estimator
kalmd     - Discrete Kalman estimator for continuous plant
lqgreg    - LQG regulator from LQ gain & Kalman estimator
```

Matlab “LQR”

```
LQR  Linear-quadratic regulator design for state space systems.

  [K,S,E] = LQR(SYS,Q,R,N) calculates the optimal
  gain matrix K such that:

   * For a continuous-time state-space model SYS, the
     state-feedback law u = -Kx  minimizes the cost function

           J = Integral {x'Qx + u'Ru + 2*x'Nu} dt

    subject to the system dynamics  dx/dt = Ax + Bu
```

Matlab “Kalman”

```
KALMAN  Continuous- or discrete-time Kalman estimator.

  [KEST,L,P] = KALMAN(SYS,QN,RN,NN) designs a Kalman
  estimator KEST for the continuous- or discrete-time
  plant with state-space model SYS. For a continuous-time model
       .
       x = Ax + Bu + Gw                  {State equation}
       y = Cx + Du + Hw + v              {Measurements}

  with known inputs u, process noise w, measurement noise v, and
  noise covariances

       E{ww'} = QN,     E{vv'} = RN,     E{wv'} = NN,

  By default, SYS is the state-space model SS(A,[B G],C,[D H])
```

Matlab “LQGREG”

```
LQGREG  Form linear-quadratic-Gaussian (LQG) regulator

  RLQG = LQGREG(KEST,K) produces an LQG regulator by
  connecting the Kalman estimator KEST designed with KALMAN
  and the state-feedback gain K designed with (D)LQR or LQRY:

        +-----------------------------+
      u |                             |
        |    +-------+                |
        +--->|       |  x_e  +-----+  |
             | KEST  |------>| -K  |---+-----> u
    y ------->|       |       +-----+
             +-------+

  The resulting regulator RLQG has input y and generates the
  commands  u = -K x_e  where x_e is the Kalman state estimate
  based on the measurements y.  This regulator should be
  connected to the plant using positive feedback.
```

Example

Consider the following system from the pole placement lecture

$$G(s) = \frac{1 + 0.5s}{s^2}$$

The following controller is suggested in Åström-Murray “Feedback Systems” p. 363

$$C(s) = 3628\frac{s + 11.02}{(s + 2)(s + 78.28)}$$

To construct an LQG controller we write the system on state space form

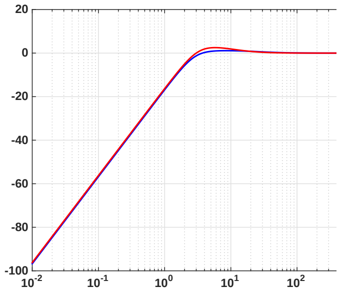
$$\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + Gw \\
y &= \begin{pmatrix} 0.5 & 1 \end{pmatrix} x + Hw + v
\end{aligned}$$

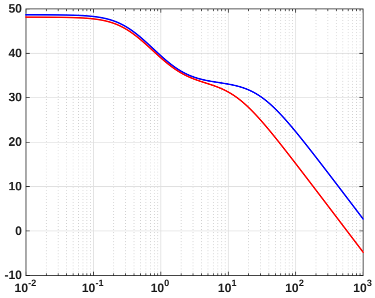
Example - slow process zero

```
A = [0 0 ; 1 0];
B = [1 ; 0];
C = [0.5 1];
D = 0;
sys = ss(A,B,C,D);
Q=diag([0 1]);
R=1e-5;
[k,s,e]=lqr(A,B,Q,R);
G=B;
H=0;
QN = 1;
RN = 1e-5;
syse = ss(A,[B G],C,[D H]);
[kest,l,p]=kalman(syse,QN,RN);
rlqg = lqgreg(kest,k);
PC = -rlqg*sys;
```

Result

$$Q_{11} = diag([0\ \ 1]),\ Q_{22} = 10^{-5},\ R_{11} = diag([0\ \ 1]),\ R_{22} = 10^{-5}$$

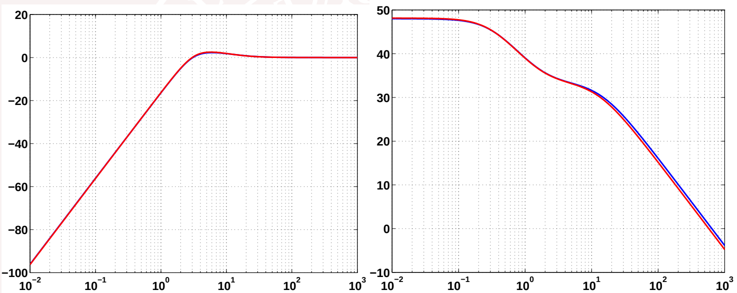




Larger high-frequency gain for LQG(blue) than default controller(red)

Result

Increased meas noise $R_{22} := 7 \cdot 10^{-5}$ and $Q_{22} = 0.7 \cdot 10^{-5}$



LQG now gives same controller as was obtained by pole-placement design earlier

Obsolete LQG Software - use at your own risk

Matlab - robust control toolbox and mutools

```
h2lqg     - continuous time H_2 synthesis.
dh2lqg    - discrete time H_2 synthesis.
normh2    - calculate H_2 norm.
lqg       - LQG optimal control synthesis.
ltru      - LQG loop transfer recovery.
ltry      - LQG loop transfer recovery.

h2syn     - H_2 control design
```

Department “LQGBOX” TFRT-7575

```
oldboxes (this might not work anymore)
[K,S] = lqrc(A,B,Q1,Q2,Q12)
[L,P] = lqec(A,C,R1,R2,R12)
kr = refc(A,B,C,D,K)
[Ac,By,Byr,Cc,Dy,Dyr] = lqgc(A,B,C,D,K,kr,L)
lqed, lqrd, refd, lqgd in discrete time
```

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Robustness

As we will see, LQ-control $u = -Lx$ automatically gives amazing robustness properties: Infinite gain margin and 60 degrees phase margin

(Warning: Only if one uses block diagonal weights !)

Critique: Rosenbrock, McMorran: Good, bad or optimal, IEEE-AC 1971. Horowitz.

Properties of LQ control

When all states can be measured, we have nice robustness properties

Loop Gain: $K(sI - A)^{-1}B$

Return Difference: $I + K(sI - A)^{-1}B$

Compare with LQG (if $D = 0$)

Loop Gain: $C(sI - A)^{-1}BK(sI - A + BK + LC)^{-1}L$

(remark on notation: $B = B_2$ on the LQ slides below)

Return Difference Formula

From Riccati equation (nice matrix exercise):

$$
\begin{aligned}
&M^T(-s)M(s) = \\
&(I + K(-sI - A^T)^{-1}B)^T D_{12}^T D_{12}(I + K(sI - A)^{-1}B)
\end{aligned}
$$

where $M(s) = D_{12} + C_1(sI - A)^{-1}B$

If no crossterms:

If $C_1^T C_1 = Q_1$, $C_1^T D_{12} = 0$ and $D_{12}^T D_{12} = Q_2$

$$
\begin{aligned}
Q_2 + B^T(-sI - A^T)^{-1}Q_1(sI - A)^{-1}B = \\
(I + K(-sI - A^T)^{-1}B)^T Q_2(I + K(sI - A)^{-1}B)
\end{aligned}
$$

This is the return difference formula for LQ

Consequences of RDF

$$(I + K(-sI - A^T)^{-1}B)^T Q_2 (I + K(sI - A)^{-1}B) \;\geq\; Q_2$$

For scalar system this becomes

$$q_2 |1 + K(sI - A)^{-1}B|^2 \geq q_2$$

therefore

$$|1 + K(sI - A)^{-1}B| \geq 1$$

$$M_s \leq 1$$

LQ Margins, Scalar case

$$M_s \leq 1$$

Disturbance rejection performance improved for all frequencies

Gain Margin $[1/2, \infty]$, Phase Margin ≥ 60 degrees.

Circle criterion: Stability under feedback with any nonlinear time-varying input gain with slopes in $(1/2, \infty)$.

Requirements: No cross-terms, $Q_{12} = 0$. All states measurable.

TAT: Why isn't this a violation of Bode's integral formula?

LQ Gain Margin, MIMO

With

$$S_i(j\omega) = (1 + K(sI - A)^{-1}B)^{-1}$$

$$\bar{\sigma}(Q_2^{1/2} S_i(j\omega) Q_2^{-1/2}) \leq 1$$

If Q_2 diagonal this gives nice MIMO gain/phase margins, see LQG course.

High Frequency Behaviour of LQ control

If $Q_{12} = 0$ then for large ω

$$K(j\omega I - A)^{-1}B \sim KB/\omega = Q_2^{-1}B^T SB/\omega$$

LQ-controller gives loop gain with roll-off 1 (unless $K = 0$)

Same conclusion for

$$K(j\omega I - A + BL)^{-1}B \sim KB/\omega = Q_2^{-1}B^T SB/\omega$$

Intution for the future: If the open loop system has roll-off larger than 1, then if one forces the LQG loop gain to approach the LQ loop gain, the LQG controller will have large high-frequency gain

Robustness of LQG

Kalman filter producing $\hat{x}$ has similar (dual) robustness properties

Since the LQG controller combines two robust parts: LQ control and Kalman filtering, it was for a long time hoped that robustness margins for the LQG controller would eventually be found

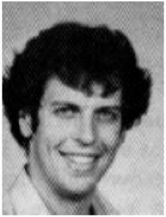
But, output feedback $u = -L\hat{x}$ was surprisingly (?) found to have no automatic guarantees for robustness

This was a dissappointment, especially for people hoping to automize design

Turned attention towards robust control, e.g. H_∞ in the 80s

A new kid on the block

Honeywell Interoffice Correspondence



Date: August 23, 1977

To: C. A. Harvey

From: J. C. Doyle

Location: S&RC, Research

cc: L. Q. Gaussian
J. A. Hauge
A. P. Kizilos
A. F. Konar
E. E. Yore
N. R. Zagalsky
Systems and Control Technology

Subject: "Guaranteed Margins for LQG Regulators"

A B S T R A C T

There aren't any.

All engineers who have been using LQG methodology may pick up their Nichols charts from the supply room.

Example - Doyle IEEE TAC 1978

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} v \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sqrt{\sigma} w
\end{aligned}
$$

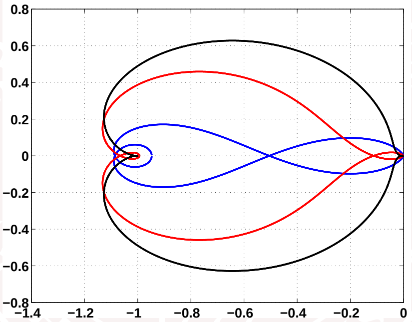
$$
\min E[(x_1 + x_2)^2 + \rho u^2]
$$

Example Doyle IEEE TAC 1978

```
A  = [1 1; 0 1];
B2 = [0 ; 1];
B1 = [1; 1];
C2 = [1 0];
C1 = [1 1];
G = B1;
H = 0*C2*B1;
sys=ss(A,B2,C2,0);
syse=ss(A,[B2 G],C2,[0 H]);
rho=1;sigma=1;
[K,S,E] = lqr(A,B2,C1'*C1,rho);
[Kest,L,P] = kalman(syse,1,sigma);
rlqg = -lqgreg(Kest,K);
loopgain = sys*rlqg;
```

Doyle's counter example

Loop gain with $\rho = \sigma = 1$ (blue), 0.01(red), 0.0001(black)



Infinitly small gain and phase margins when ρ and σ become small

The symmetric root locus

Assume $Q_{11} = C^T C, Q_{12} = 0, Q_{22} = \rho I$, then for SISO systems

$$\begin{aligned} G(s) &= C(sI - A)^{-1} B =: \frac{B(s)}{A(s)} \\ I + H(s) &:= I + K(sI - A)^{-1} B =: \frac{P(s)}{A(s)} \end{aligned}$$

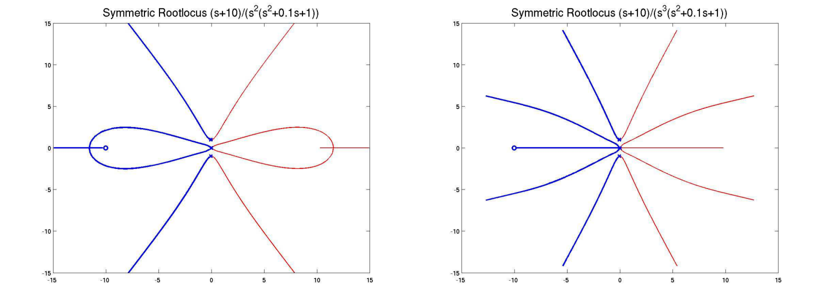
Closed loop characteristic equation $P(s) = 0$ (TAT: Why?)

Riccati equation gives (return difference formula)

$$\begin{aligned} Q_2 + G(-s)G(s) &= (I + H(-s))Q_2(I + H(s)) \\ \rho A(-s)A(s) + B(-s)B(s) &= \rho P(-s)P(s) \end{aligned}$$

`symlocc`, `symlocd` in matlab (oldboxes)

Symmetric Root Locus



Symmetric root loci for $G(s) = \frac{s+10}{s^2(s^2+0.1s+1)}$ and $G(s)/s$.

```
oldboxes;robotdata
[b,a]=tfdata(sys2);b=b{1};a=a{1};
locus=symlocc(b,a,1e-6,1e10,0.003);
plot(locus(:,2:end),'b','Linewidth',2)
```

Cheap control $\rho \to 0$

$$\rho A(-s)A(s) + B(-s)B(s) = \rho P(-s)P(s)$$

Eigenvalues of closed loop tend to stable zeros of $B(-s)B(s)$ and the rest tend to ∞ as stable roots of

$$s^{2d} = \text{const} \cdot \rho$$

An interesting formula - cheap control

$$\min \int_0^\infty |y(t) - 1|^2 dt = 2 \sum_{Re z_j > 0} \mathrm{Re} \frac{1}{z_j}$$

where the sum is over all non-minimum phase zeros.

Reference: Qui-Davison, Automatica 1993 pp. 337-349

TAT: Where have you seen something similar before?

Expensive control $\rho \to \infty$

$$\rho A(-s)A(s) + B(-s)B(s) = \rho P(-s)P(s)$$

Eigenvalues of closed loop tend to stable zeros of $A(-s)A(s)$

Example

$$\min u^2, \quad \dot{x} = x + u$$

$A(s) = s - 1$ unstable.

Optimal controller $u = -2x$ gives

$$\dot{x} = -x$$

$P(s) = s + 1$

Energy minimizing stabilization

Consider the system

$$\dot{x} = Ax + B(u + w), \quad u = -Kx$$

where w is unitary white noise.

The minimal control effort needed to stabilize the system is

$$\min E|u|^2 = 2 \sum_{\mathrm{Re}\, p > 0} \mathrm{Re}\; p$$

where the sum is over all unstable poles (exercise).

The optimal closed loop system $A - BK$ has eigenvalues in the open loop stable poles and the mirror image of the open loop unstable poles.

“The cheapest way to stabilize an unstable pole is to mirror it”

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How to tune the weights

Having a state-space realization where the states have a physical meaning aids the intuition

It is helpful to choose scalings so that all interesting signal levels are roughly the same size

Q_1 incr, or Q_2 decr. gives faster control

R_1 incr, or R_2 decr. gives faster observer

(Bryson's) rule of thumb

$$
\begin{aligned}
Q_1 &= \text{diag}(\alpha_1, \ldots, \alpha_n) \\
Q_2 &= \text{diag}(\beta_1, \ldots, \beta_m)
\end{aligned}
$$

Let $\alpha_i \sim (x_i)^{-2}$ and $\beta_i \sim (u_i)^{-2}$ where x_i and u_i denote allowable sizes on state i and input i

Similar intuition for the noise weights R_1 and R_2.

Note that multiplying all elements of Q by the same factor does not change the controller. Similar for the R matrices.

Tuning Tricks

Introducing an extra punishment of

$$(\dot{x}_i + \alpha x_i)^2$$

should move the system closer to $\dot{x}_i = -\alpha x_i$.

Gives cross-terms

Another tuning trick

$$G(s) = \frac{1}{(s+1)(s^2+1)}$$

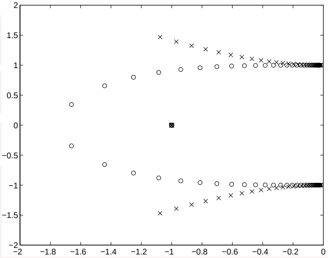
Want to increase damping without moving the pole in $s = -1$.

This can be achieved by weights that are zero on the eigendirections to $s = -1$.

Example -continued

```
s =
   0.7071             0.7071             0.4082
        0 + 0.7071i        0 - 0.7071i  -0.4082
        0                  0             0.8165
d =
        0 + 1.0000i        0                  0
        0                  0 - 1.0000i        0
        0                  0            -1.0000
```

$$Q_1 = q_i q_i^T, \qquad q_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$



Example Aircraft - wind gust turbulence

Taken from Anderson-Moore Optimal Control Linear Quadratic Methods, pp.222-224

6 state model of aircraft subject to wind gust turbulence

$$\dot{x} = Ax + Bu + B_v v, \quad y = Cx + w$$

Two outputs y_f and y_a forward and aft accelerations, one input

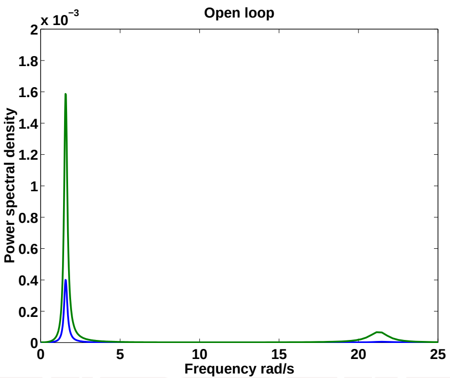
Open loop resonances at 1.6 and 21 rad/s

$$A = \begin{pmatrix} -3\cdot 10^{-8} & & & 0 \\ & -7\cdot 10^{-3} & & \\ & & \begin{pmatrix} -0.12 & 1.57 \\ -1.57 & -0.12 \end{pmatrix} & \\ 0 & & & \begin{pmatrix} -0.71 & 21.3 \\ -21.3 & -0.71 \end{pmatrix} \end{pmatrix}$$

See home page for full model

Open Loop

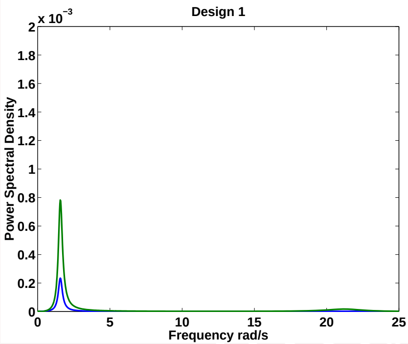
Turbulence without a controller

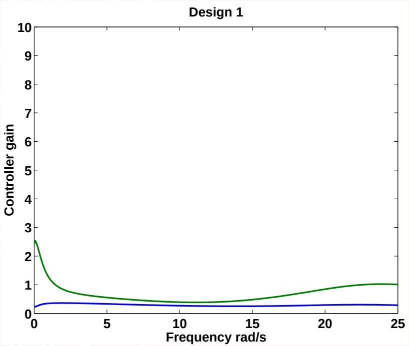


y_f (blue) and y_a (green)

Design 1

$$\min E[y_f^2 + y_a^2 + 0.2u^2]$$



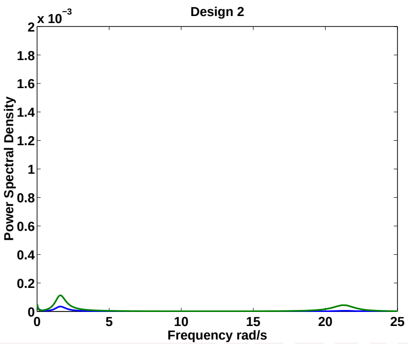


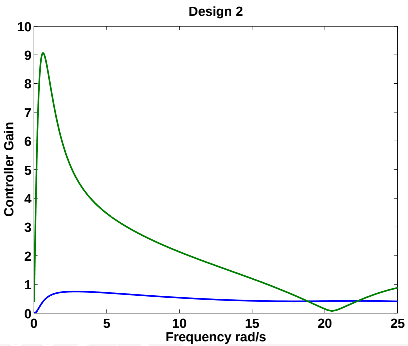
Want to increase damping of resonance at 1.5 rad/s

Penalise x_3 and x_4 more

Design 2

$$\min E[y_f^2 + y_a^2 + 4x_3^2 + 4x_4^2 + u^2]$$

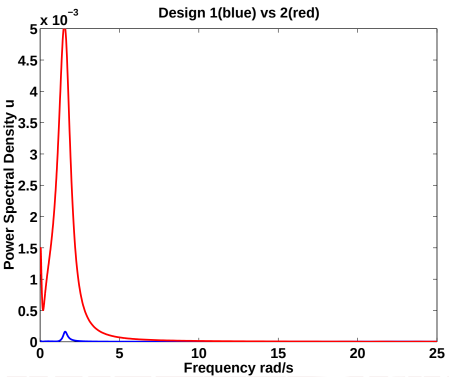




Big improvement at 1.5rad/s

But is Andersson-Moore's Design 2 any good?

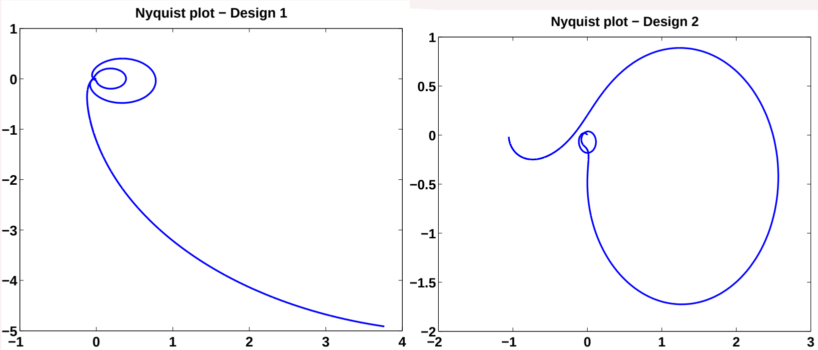
Comparison Design 1 vs 2



Design 2 has much more control effort around 1.5 rad/s

But that's perhaps ok. But how about robustness?

Comparison Design 1 vs 2



Andersson-Moore's design 2 has very bad robustness margins, e.g. $M_s \sim M_t \sim 20$.

A change in process gain of 5 % gives an unstable loop

Conclusion: Even the masters can make a bad design with LQG. No guarantees for robustness.

Lecture - LQG Design

- Introduction
- The H_2-norm
- Formula for the optimal LQG controller
- Software, Examples
- Properties of the LQ and LQG controller
- Design tricks, how to tune the knobs

Next lecture

- What do the “technical conditions” mean?
- How to get integral action etc
- Loop Transfer Recovery (LTR)
- More Examples