Exercise 6 LQG and H_∞

1. Use the appropriate Riccati equation to prove the Kalman filter identity

$$\begin{aligned} &R_2 + C_2(sI - A)^{-1}R_1(-sI - A^T)^{-1}C_2^T \\ &\quad = [I_p + C_2(sI - A)^{-1}L]R_2[I_p + C_2(-sI - A^T)^{-1}L]^T \end{aligned}$$

Use duality to deduce the return difference formula

$$\begin{aligned} &Q_2 + B^T(-sI - A^T)^{-1}Q_1(sI - A)^{-1}B = \\ &\quad [I_m + K(-sI - A^T)^{-1}B]^T Q_2[I_m + K(sI - A)^{-1}B] \end{aligned}$$

2. Consider the Doyle-Stein LTR example from the LQG lecture

$$G(s) = \frac{s+2}{(s+1)(s+3)}$$

See the slides, or their article, for more details.

a. Evaluate the H_2-norm for the system from v to z where $z^Tz = x^TQ_1x + u^TQ_2u$ and the maximum sensitivity, M_S, of the closed loop system for $q = 0$.

b. Plot H_2-norm versus M_S for varying values of q. Is much H_2-optimality lost to obtain robustness?

3. Consider the Rosenbrock example from the Interaction lecture

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

which has a multivariable zero in $s = 1$. Design a controller using LQG and try to achieve a gain crossover frequency larger than $w_{gc} = 0.5$ rad/s that has reasonable robustness.

4. The file `quadtank.m` on the home page contains a linear model of a symmetric quadtank, with outputs being the two lower tanks. The parameter setting $\gamma = 0.3$ corresponds to a non-minimum phase system which is difficult to control.

Make an LQG design with reasonable performance which has integral action using either

a. explicit integration augmenting the system with integrator states $\dot{x}_i = y - r \in R^2$ (no estimation of these states should be done in the Kalman filter)

b. augmenting the system with a constant input disturbance model, i.e. $\dot{x} = Ax + Bu + Bd$ where $\dot{d} = 0$ and $u = -K\hat{x} - \hat{d}$.

In both cases, plot the gain (singular values vs frequency) of the resulting controller and the GOF.

Also verify that the step responses of the closed loop systems look reasonable.

5. Prove the formulas mention on the Robust Control lecture

$$\begin{aligned} \text{gain margin} &\geq \frac{1+b_{P,K}}{1-b_{P,K}}, \\ \text{phase margin} &\geq 2\arcsin(b_{P,K}). \end{aligned}$$

6. Find a rational controller $C(s)$ that stabilizes both $P(s) = \frac{1}{s}$ and $P(s) = -\frac{1}{s}$ or prove that it is impossible.

7. Calculate the nu-gap δ_ν, for varying parameters a, between the processes

a.

$$G_1(s) = \frac{1}{s+a} \text{ and } G_2(s) = \frac{1}{s-a},$$

b.

$$G_1(s) = \frac{1}{s+a} \text{ and } G_2(s) = \frac{1}{a-s},$$

c.

$$G_1(s) = \frac{1}{s+1} \text{ and } G_2(s) = \frac{a}{s+a}$$

d.

$$G_1(s) = \frac{1}{s+1} \text{ and } G_2(s) = \frac{1}{(s+1)^2}$$

e.

$$G_1(s) = \frac{1}{s-1} \text{ and } G_2(s) = \frac{1}{(s-1)^2}$$

f.

$$G_1(s) = \frac{1}{s-1} \text{ and } G_2(s) = \frac{1}{(s-1)(s+1)}$$

Hint: Use matlab-command `gapmetric`.

8. The solution to the H_∞ problem presented at the lecture (implemented in the matlab-routine `hinfsyn`) solves the so called sub-optimal problem. Given γ, determine if a controller exists giving a closed loop with

$$\|T_{zw}\|_\infty < \gamma.$$

The optimal level, γ_c, can then be found by decreasing γ until no solution exists. To study the optimal H_∞ controller, consider the system given in the matlab-file `goldenratioex.m`. The system describes the feedforward optimization problem

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -x_2 + x_1 + w \\ z &= \begin{bmatrix} x_2/\rho \\ u \end{bmatrix} \\ y &= w \end{aligned}$$

a. For the case $\rho = 1$, use `hinfsyn` to find the optimal controller $u = K(s)w$ when $\gamma = \gamma_c$. Compare with the analytical solution $K(s) = (s+1)/(k(s+1)+1)$ with $k = 1.3953$. Hint: The optimal value is $\gamma_c \approx 0.7167$.

b. Do the same for $\rho = \sqrt{2}$. What order will the controller given by `hinfsyn` be now? Hint: The optimal value is $\gamma_c = 1/\sqrt{3}$.

For interested: More details can be found in an article by Bernhardsson and Hagander from 1990.

9. Use `mixsyn` to do control of the motor

$$G(s) = \frac{20}{s(s+1)}$$

achieving $|\omega S(j\omega)|_\infty < k_1$ and $|C(j\omega)S(j\omega)|_\infty < k_2$. Plot the region in the (k_1, k_2)-plane that you were able to achieve. Hint: You can use the file `motorex6.m` as a start.

10. Use the file `aircglover.m` to do Glover-MacFarlane design with loopsyn on the aircraft example. In the design, the 3 PI controller had the same parameters. Redo the design and try to reduce the control peak due to a change of height reference (first input, i.e. first column in figure 6) while maintaining a settling time of 1 second, good damping and mainly diagonal (noninteracting) response.