Lecture 7

- Theory for polynomial matrices
- Hermite and Smith normal forms
- Smith McMillan form
- Poles and Zeros

Rugh Ch 16-17 (can skip proofs of 16.7,17.4,17.5,17.6)

Polynomial matrix fraction descriptions

There are two natural generalisation to the SISO description

$$G(s) = \frac{n(s)}{d(s)}$$

Right polyomial matrix fraction description:

$$G(s) = N_R(s)D_R(s)^{-1} \qquad \begin{cases} D_R(s)X(s) = U(s) \\ Y(s) = N_R(s)X(s) \end{cases}$$

Left polynomial matrix fraction description

$$G(s) = D_L(s)^{-1}N_L(s) \qquad D_L(s)Y = N_L(s)U$$

where N_R, D_R, N_L, D_L are polynomial matrices

Left and Right MFDs - example

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{2}{s+1} \end{pmatrix}$$

Right MFD $\quad G(s) = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 0 & s+1 \end{pmatrix}^{-1}$

Left MFD $\quad G(s) = ((s+2)(s+1))^{-1} \begin{pmatrix} s+1 & 2(s+2) \end{pmatrix}$

Note that the dimensions of D_R and D_L are not the same

Note however that

$$\det\, D_R(s) = \det\, D_L(s) = (s+2)(s+1)$$

Questions

What properties can be seen from $D(s)$ and $N(s)$?

What are the poles and zeros?

Example: The MIMO system

$$G(s) = \begin{pmatrix} \frac{s+1}{s+2} & 0 \\ 0 & \frac{s+2}{s+1} \end{pmatrix}$$

has poles in $-1, -2$ and zeros in $-1, -2$ (but in different "directions")

Note however that $\det G(s) \equiv 1$

How to cancel "common factors" in N and D?

Common Factors of $N(s)$ and $D(s)$

$R(s)$ is said to be a **common right divisor** if $\exists\, \widetilde{N}(s), \widetilde{D}(s)$

$$\begin{pmatrix} N(s) \\ D(s) \end{pmatrix} = \begin{pmatrix} \widetilde{N}(s) \\ \widetilde{D}(s) \end{pmatrix} R(s)$$

- $N(s)D^{-1}(s) = \widetilde{N}(s)\widetilde{D}^{-1}(s)$
- If $R(s)$ can be written $R(s) = S(s)\widetilde{R}(s)$ for every crd $\widetilde{R}(s)$, then $R(s)$ is a **greatest common right divisor (gcrd)**
- A polynomial matrix whose inverse is also polynomial is a trivial common factor. Such matrix is called "**unimodular**".
- If a gcrd of N and D is unimodular then N and D are said to be **"right coprime"**
- Common left divisor, gcld, left coprime are defined analogously for left MFDs

Unimodular Matrices

$A(s)$ unimodular $\quad \Leftrightarrow \quad \det\, A(s)$ is a nonzero constant

Proof:
If there is $B(s)$ with $A(s)B(s) = B(s)A(s) = I$, then $\det A(s) \cdot \det B(s) = 1$ and both $A(s)$ and $B(s)$ have constant nonzero determinants.

If $A(s)$ has constant nonzero determinant then

$$A(s)\mathrm{adj}A(s) = \det A(s) I = cI \neq 0$$

and hence $A^{-1}(s) = \mathrm{adj}A(s)/c$ which is a polynomial matrix, hence $A(s)$ is unimodular

Unimodular Matrices

Examples of unimodular matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & a(s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When multiplying a matrix from the left they correspond to

- exchange of first two rows
- addition of $a(s)$ times second row to first row
- multiplication of first row by a

Elementary row (column) operations = unimodular matrix left (right) multiplication

Hermite Form - row operations version

For a polynomial matrix $P(s)$ with independent columns it is possible to find a unimodular matrix $U(s)$ (row operations) so

$$U(s)P(s) = \begin{pmatrix} \times & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \times \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where diagonal elements are

- nonzero, monic polynomials
- of higher degree than elements in the same column

Hermite Form - column operation version

For a matrix with independent rows, an analogous lower triangular form can be obtained by multiplying from the right with a unimodular matrix (e.g. by column operations)

$$P(s)C(s) = \begin{pmatrix} \times & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \times & \times & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & & \vdots \\ \times & \times & \cdots & \times & 0 & \cdots & 0 \end{pmatrix}$$

Proof of Hermite form: Iterative constructive proof, similar to Gauss elimination, but using “polynomial division with remainder” as basic step instead of division

Hermite Form in Maple

```
with(LinearAlgebra); with(MatrixPolynomialAlgebra);
G:= Matrix(4,2,[s^2+3*s+2,0,0,s^2+3*s+2,s+2,1,s,2*s+1]);
H := HermiteForm(G,s);
latex(G);latex(H);
```

$$G = \begin{pmatrix} s^2+3s+2 & 0 \\ 0 & s^2+3s+2 \\ s+2 & 1 \\ 2 & 2s+1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 1 \\ 0 & s+1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Finding common factors and computing a gcrd

Given $G(s) = N_R(s)D_R^{-1}(s)$, use Hermite to get unimodular U:

$$\begin{pmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{pmatrix} \begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix} = \begin{pmatrix} R(s) \\ 0 \end{pmatrix}$$

With $V = U^{-1}$ we get

$$\begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix} = \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix} \begin{pmatrix} R(s) \\ 0 \end{pmatrix}$$

- R is a gcrd of N_R and D_R
- V_{11} is nonsing., $\det V_{11} = \mathrm{const} \cdot \det U_{22}$
- $G(s) = V_{21}(s)V_{11}^{-1}(s)$ right coprime MFD
- $G(s) = -U_{22}(s)^{-1}U_{21}(s)$ left coprime MFD

3 min Exercise

Write down the dual result if we instead have a left MFD

$$G(s) = D_L^{-1}(s)N_L(s)$$

The audience is thinking

Polynomial Maple Toolbox

```
with(LinearAlgebra):
with(MatrixPolynomialAlgebra):

List of MatrixPolynomialAlgebra Package Commands

Coeff   ColumnReducedForm  Degree       HermiteForm Lcoeff
Ldegree     LeftDivision   MahlerSystem MatrixGCLD  MatrixGCRD
MatrixLCLM  MatrixLCRM     MinimalBasis PopovForm   RightDivision
RowReducedForm SmithForm   Tcoeff
```

Example

$$G(s) = \begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & \frac{s}{(s+2)^2} \\ -\frac{s}{(s+2)^2} & -\frac{s}{(s+2)^2} \end{bmatrix} =$$

$$= \begin{pmatrix} s & s \\ -s(s+1)^2 & -s \end{pmatrix} \begin{pmatrix} (s+1)^2(s+2)^2 & 0 \\ 0 & (s+2)^2 \end{pmatrix}^{-1} = N_R D_R^{-1}$$

Find common factors and compute left MFD and right MFD

$$P(s) := \begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix} = \begin{pmatrix} (s+1)^2(s+2)^2 & 0 \\ 0 & (s+2)^2 \\ s & s \\ -s(s+1)^2 & -s \end{pmatrix}$$

$$U(s) \begin{pmatrix} P(s) & I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} R(s) \\ 0 \end{pmatrix} & U(s) \end{pmatrix}$$

Example - continued

```
with(MatrixPolynomialAlgebra): with(LinearAlgebra):with(linalg):
P:=Matrix([[(s+1)^2*(s+2)^2,0],[0,(s+2)^2],[s,s],[-s*(s+1)^2,-s]]);
PI:=convert(augment(P,IdentityMatrix(4,4)),Matrix):
RU:=map(factor,HermiteForm(PI,s)):
R:=submatrix(RU,1..2,1..2);
U:=submatrix(RU,1..4,3..6):
V:=map(factor,inverse(U)):
V11:=submatrix(V,1..2,1..2);V21:=submatrix(V,3..4,1..2);
U21:=submatrix(U,3..4,1..2);U22:=submatrix(U,3..4,3..4);
latex(R);latex(U21);latex(U22);latex(V11); latex(V21);
```

$$RU = \begin{bmatrix} 1 & 1 & 1/4 & 1/4+s/2 & -s^2/2-2\,s-5/2 & s/4+1/2 \\ 0 & s+2 & 0 & -s/4+1/2 & \left(s+1\right)^2/4 & 1/4 \\ 0 & 0 & s & s & 0 & \left(s+2\right)^2 \\ 0 & 0 & 0 & s^2 & -\left(s+2\right)\left(s+1\right)^2 & -s-2 \end{bmatrix}$$

Example - continued

Common factor

$$R = \begin{bmatrix} 1 & 1 \\ 0 & s+2 \end{bmatrix}$$

Submatrices of U and V needed:

$$U_{22} = \begin{bmatrix} 0 & (s+2)^2 \\ -(s+2)(s+1)^2 & -s-2 \end{bmatrix}, \; U_{21} = \begin{bmatrix} s & s \\ 0 & s^2 \end{bmatrix}$$

$$V_{21} = \begin{bmatrix} s & 0 \\ -s(s+1)^2 & s^2 \end{bmatrix}, V_{11} = \begin{bmatrix} (s+1)^2(s+2)^2 & -(s+2)(s+1)^2 \\ 0 & s+2 \end{bmatrix}$$

Example - continued

Right coprime MFD of $G(s) = V_{21}(s)V_{11}^{-1}(s)$:

$$N_R D_R^{-1} = \begin{bmatrix} s & 0 \\ -s\left(s+1\right)^2 & s^2 \end{bmatrix} \begin{bmatrix} \left(s+1\right)^2\left(s+2\right)^2 & -\left(s+2\right)\left(s+1\right)^2 \\ 0 & s+2 \end{bmatrix}^{-1}$$

Left coprime MFD of $G(s) = -U_{22}^{-1}(s)U_{21}(s)$:

$$D_L^{-1} N_L = \begin{bmatrix} 0 & -\left(s+2\right)^2 \\ \left(s+2\right)\left(s+1\right)^2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} s & s \\ 0 & s^2 \end{bmatrix}$$

Note that

$$\begin{aligned} \det\, D_R(s) &= \det\, D_L(s) = (s+1)^2(s+2)^3 \\ \det\, N_R(s) &= \det\, N_L(s) = s^3 \end{aligned}$$

A useful result

Assume $P(s)$ and $Q(s)$ have the same number of columns, n. The following are then equivalent (left version also exists)

- $P(s)$ and $Q(s)$ are right coprime
- There exists polynomial matrices $X(s)$ and $Y(s)$ so (Bezout identity)

$$X(s)P(s) + Y(s)Q(s) = I_n$$

- For every complex s

$$\operatorname{rank} \begin{pmatrix} Q(s) \\ P(s) \end{pmatrix} = n$$

Proof: Follows directly from the Hermite form.

Coprime MFDs are (almost) unique

Theorem If we have two coprime right MFDs

$$G(s) = N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s)$$

then there is a unimodular matrix $U(s)$ such that

$$N_1(s) = N_2(s)U(s), \qquad D_1(s) = D_2(s)U(s)$$

Remark: As a consequence $\det D_1(s) = k \det D_2(s),\ \ k \neq 0$

An analogous result of course holds for left coprime MFDs

Comparing left and right MFDs

Theorem If

$$G(s) = D_L^{-1}(s) N_L(s) = N_R(s) D_R^{-1}(s)$$

with both MFDs coprime, then

$$\det D_L(s) = k \det D_R(s), \;\; k \neq 0$$

The degree of $D(s)$ in any coprime MFD is called the **McMillan degree** of $G(s)$. This degree equals the dimension of any minimal representation of $G(s)$

To show this, and to find a state space realisation, one more property of MFDs is studied in Rugh: “column reduced” (right MFD), or “row reduced” (left MFD). We will skip the proof of these results (e.g. Rugh 17.4).

An Observation

The left MFD $(sI-A)^{-1}B$ is coprime $\Leftrightarrow$ $\{A, B\}$ is controllable

The right MFD $C(sI-A)^{-1}$ is coprime $\Leftrightarrow$ $\{A, C\}$ is observable

Smith Form and equivalence

By simultaneous row and column operations we can go beyond the Hermite form and obtain a diagonal form

The poles and zeros of the systems can then be seen clearly

Two polynomial matrices $A(s)$ and $B(s)$ are "equivalent" if $A(s)$ can be transformed into $B(s)$ using elementary row and column operations. We then write

$$A(s) \sim B(s)$$

Remark: $A(s) \sim B(s)$ if and only if there exist $P(s)$ and $Q(s)$ such that $B(s) = P(s)A(s)Q(s)$ where $P(s)$ and $Q(s)$ are products of elementary matrices, i.e. unimodular matrices

Theorem - Smith Normal Form

For any polynomial matrix $A(s)$ it holds that

$$A(s) \sim \begin{bmatrix} D_r(s) & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$D_r(s) = \mathrm{diag}(i_1(s), i_2(s), \ldots, i_r(s))$$

and where

- $i_k(s)$ are monic polynomials
- i_k divides i_{k+1} for $k = 1, 2, \ldots, r-1$.

Definition : $i_k(s)$, $k = 1, 2, \ldots, r$ are called the **invariant polynomials** of $A(s)$.

Example - Maple

```
with(MatrixPolynomialAlgebra):
A:=Matrix([[s+2,1],[s,2*s+1]]);
                 [s + 2       1    ]
           A := [                  ]
                 [  s      2 s + 1]

> SmithForm(A);
                 [1         0      ]
                 [                  ]
                 [             2    ]
                 [0    2 s + s   + 1]

> latex(map(factor,SmithForm(A)));
```

$$\begin{bmatrix} 1 & 0 \\ 0 & (s+1)^2 \end{bmatrix}$$

Definition: Determinantal divisors

A **determinantal divisor** $d_j(s)$ of a polynomial matrix $A(s)$ is the greatest common divisor of all the minors of order j in $A(s)$, $j = 1, 2, \dots, \min(m, n)$.

$d_1(s) =$ GCD of all elements

$d_2(s) =$ GCD of all 2×2 subdeterminants

etc

$d_n(s) =$ const · determinant of $A(s)$

where the constant is chosen so d_n becomes monic.

Lemma

The determinantal divisors are invariant under elementary operations.

Proof: Let $B(s) = P(s)A(s)$ where $P(s)$ is unimodular. By the Cauchy-Binet formula for determinants

$$\det(B[I,J](s)) = \sum_{\#K=j} \det(P[I,K](s))\det(A[K,J](s))$$

where $\#I = \#J = j$. It follows that $A(s)$ and $B(s)$ have the same determinantal divisors (think).

Theorem

The Smith form is unique, and can be found from the determinantal divisors

Proof: A matrix

$$\begin{bmatrix} i_1(s) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & i_2(s) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & & i_r(s) & & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

where i_k divides i_{k+1} for $k = 1, 2, \ldots, r-1$ has determinantal divisors given by

Proof continued

$$d_m(s) = i_1(s) i_2(s) \cdots i_m(s),\ m = 1, 2, \ldots, r$$
$$d_m(s) = 0,\ m > r$$

Hence

$$i_1(s) = d_1(s)$$
$$i_m(s) = d_m(s)/d_{m-1}(s),\ 2 \leq m \leq r$$

Since the determinantal divisors are invariant under elementary operations, $i_k(s)$ are uniquely determined by the original matrix.

Example

Consider the earlier example

$$A(s) = \begin{pmatrix} s+2 & 1 \\ s & 2s+1 \end{pmatrix}$$

The determinantal divisors are

d_1: GCD of $(s+2), 1, s, (2s+1)$, i.e. $d_1 = 1$

d_2: $\det A(s) = (s+2)(2s+1) - s = 2(s+1)^2$, i.e. $d_2 = (s+1)^2$

Hence the Smith form is (as already computed by Maple)

$$A(s) \sim \begin{pmatrix} 1 & 0 \\ 0 & (s+1)^2 \end{pmatrix}$$

Theorem

Two polynomial matrices of the same order are equivalent if and only if they have the same invariant polynomials

Proof: Use elementary operations to bring both matrices to their Smith form. The result follows from the uniqueness of the Smith form.

The Smith McMillan Form

Let $d(s)$ be the least common multiple of denominators and write

$$G(s) = \frac{1}{d(s)} N(s)$$

Find Smith form of $N(s) = P(s)\Lambda(s)Q(s)$, P, Q unimodular

The Smith McMillan form is then

$$G(s) = P(s) \begin{pmatrix} \text{diag}\left(\frac{\epsilon_i(s)}{\psi_i(s)}\right) & 0 \\ 0 & 0 \end{pmatrix} Q(s)$$

where ϵ_i, ψ_i without common factors

$$\frac{\epsilon_i(s)}{\psi_i(s)} = \frac{\lambda_i(s)}{d(s)}, \quad \psi_{i+1}(s)|\psi_i(s), \;\; \epsilon_i(s)|\epsilon_{i+1}(s), \;\; \psi_1(s) = d(s)$$

Poles and Zeros

Using the Smith McMillan form we define

- The roots of $\epsilon_i(s)$ are the (transmission) zeros
- The roots of $\psi_i(s)$ are the poles

(counted with multiplicities)

Example

$$\begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} \end{pmatrix} = \frac{1}{(s+1)(s+2)} \begin{pmatrix} s+2 & 1 \\ s & 2s+1 \end{pmatrix}$$

The Smith McMillan form is

$$\frac{1}{(s+1)(s+2)} \begin{pmatrix} 1 & 0 \\ 0 & (s+1)^2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{(s+1)(s+2)} & 0 \\ 0 & \dfrac{s+1}{s+2} \end{pmatrix}$$

with

$$\epsilon_1 = 1, \epsilon_2 = s+1; \;\; \psi_1 = (s+1)(s+2), \psi_2 = s+2$$

zeros: -1
poles: $-1, -2, -2$

Another Example

Consider a system of the form

$$G(s) = \begin{pmatrix} \frac{b_1(s)}{a_1(s)} & \frac{b_2(s)}{a_2(s)} \\ 0 & \frac{b_3(s)}{a_3(s)} \end{pmatrix}$$

where $b_1, b_2, b_3, a_1, a_2, a_3$ have no common factors.

$$G(s) = \frac{1}{a_1(s)a_2(s)a_3(s)} \begin{pmatrix} b_1(s)a_2(s)a_3(s) & b_2(s)a_1(s)a_3(s) \\ 0 & b_3(s)a_1(s)a_2(s) \end{pmatrix}$$

The invariant factors are

$$\begin{aligned} i_1(s) &= 1 \\ i_2(s) &= b_1(s)b_3(s)a_1(s)a_2^2(s)a_3(s) \end{aligned}$$

Example

The Smith-McMillan form of $G(s)$ is hence

$$\frac{1}{a_1(s)a_2(s)a_3(s)} \begin{pmatrix} 1 & 0 \\ 0 & b_1(s)b_3(s)a_1(s)a_2^2(s)a_3(s) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{a_1(s)a_2(s)a_3(s)} & 0 \\ 0 & b_1(s)b_3(s)a_2(s) \end{pmatrix}$$

Poles: Roots of $a_1(s)a_2(s)a_3(s)$

Zeros: Roots of $b_1(s)b_3(s)a_2(s)$

Roots of $a_2(s)$ are both poles and zeros of the system!

Invariants in transfer functions

Let $G(s)$ have different left or right coprime MFDs. Then it can be seen that

- All numerator matrices $N(s)$ have the same Smith form
- All denominator matrices $D(s)$ have the same Smith form (except for extra 1s on the diagonal)
- The invariant polynomials of the numerators matrices are the $\epsilon_i(s)$ of the SmithMcMillan form of G
- The invariant polynomials of the denominator matrices are the $\psi_i(s)$ of the SmithMcMillan form of G
- The zeros are the s-values for which the rank of $N(s)$ drops below its normal rank
- The poles are the roots of $\det D(s) = 0$