Session 6

Least squares problems. Adjoint operator.

Reading Assignment

Luenberger "Optimization by Vector Space Methods" Chapter 6.
The link for the book PDF:
`https://www.dropbox.com/s/82czmjit8ic4oay/David%20G.%20Luenberger%20-%20Optimization%20by%20Vector%20Space%20Methods.pdf?dl=0`

Exercise 6.1 [Reading] Least squares solution to linear equations also has a clear interpretation by using projection matrices, and singular value decomposition.

Here is a book chapter for further reading:
`https://www.dropbox.com/s/j47msteck1ywupe/Least%20squares-Singular%20Values.pdf?dl=0`

Excerpt from
`https://www.math.ucdavis.edu/~linear/linear-guest.pdf`

Exercise 6.2 Find the minimal (squared) energy $\int_0^1 |u(t)|^2 dt$ of an input signal u such that $\dot{x}(t) = -x(t) + u(t)$ and

- a) $x(0) = 0$, $x(1) = 1$
- b) $x(0) = 1$, $x(1) = 0$,
- c) $x(0) = 1$, $x(1) = 1$.

Exercise 6.3 Show the relation mentioned on the lecture

$$[\Phi_A(s,t)]^* = \Phi_{-A^T}(t,s)$$

Exercise 6.4 Prove for a matrix A that $\mathcal{R}(A) = \mathcal{R}(AA^*)$

Exercise 6.5 Show that if L is invertible then

$$(L^*)^{-1} = (L^{-1})^*$$

Exercise 6.6 Assume L^*L is invertible. Prove that the minimum for the Least Squares Problem 1 is given by

$$|Lu - v|^2 = < v, (I - P_L)v >$$

where

$$P_L = L(L^*L)^{-1}L^*$$

Also show that P_L is an orthogonal projection operator, i.e. $P_L^2 = P_L = P_L^*$.

Exercise 6.7 Derive the formula of least norm control u by the controllability Grammian matrix in Page 26 of the lecture slides.

Exercise 6.8 Formulate observability and reconstructability using operators, i.e. the possibility to find $x(t_0)$ and $x(t_1)$ given the output and the input during the interval $[t_0, t_1]$. The nullspaces of the corresponding operators are now the important spaces. (Hint: Use

$$\begin{aligned} y(t) &= (L_1 x_0 + L_2 u_{[t_0,t_1]})(t) \\ &= C(t)\Phi(t,t_0)x_0 + C(t)\int_{t_0}^{t} \Phi(t,s)B(s)u(s)\,ds \end{aligned}$$

and discuss the solvability of the equation $L_1 x_0 = b$. Describe L_1^* and use $\mathcal{N}(L_1) = \mathcal{N}(L_1^* L_1)$.)

Hand in problem - to be handed in at the exercise session

Exercise 6.9 For the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2 + u_1 \\ \dot{x}_2 &= -2x_2 + u_1 + u_2 \end{aligned}$$

determine u_1 and u_2 for $t \in [0, 1]$ to bring (x_1, x_2) from $(1, 0)$ at $t = 0$ to $(0, 0)$ at $t = 1$ while minimizing $\int_0^1 \|u(t)\|^2 dt$.

Exercise 6.10 The LTV system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad x(t_0) = 0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

gives a linear map $y = Lu : L_2[t_0, t_1] \to L_2[t_0, t_1]$. Find an LTV system describing the adjoint L^*.