Session 2

Transition matrix properties. Change of coordinates. Periodic Systems.

Reading Assignment

Rugh (1996 edition) Rugh Chapters 4-5 (and scan Chapters 20-21).

Exercise 2.1 = Rugh 4.1

Exercise 2.2 = Rugh 4.3

Exercise 2.3 = Rugh 4.4

Exercise 2.4 = Rugh 4.6

Exercise 2.5 = Rugh 4.9 (you don't have to use the hint). Note the relation between skew symmetric matrix and orthogonal matrix.

Exercise 2.6 = Rugh Rugh 5.14

Exercise 2.7 = Rugh Rugh 5.19

Exercise 2.8 In the lecture we show that in general, for time-varying $A(t)$,

$$\Phi(t, t_0) \neq \exp\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$$

unless some commuting conditions are satisfied. Consider a time-varying matrix $A(t) \in \mathbb{R}^{m\times m}$ and the integral $M(t, t_0) := \exp\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$. If one of the following conditions holds, then $A(t)$ and $M(t, t_0)$ commute.

- $A(t) = A$ is a constant matrix (shown in the book).
- $A(t) = \alpha(t)B$ where $\alpha(t)$ is a scalar time-varying function and B is a constant matrix (shown in the book).
- $A(t) = \sum_{i=1}^{n} \alpha_i(t)B_i$, where each $\alpha_i(t)$ is a scalar time-varying function and each B_i is a constant matrix which commutes with each other; i.e., $B_iB_j = B_jB_i$, $\forall i, j \in \{1, 2, \cdots, n\}$.
- There exists a factorization $A(t) = TD(t)T^{-1}$, where D is a diagonal matrix $D(t) = \text{diag}\{d_1(t), d_2(t), \cdots, d_m(t)\}$

Try to prove the above conditions.

Exercise 2.9 = Rugh 20.10

Exercise 2.10 = Rugh 20.11

Exercise 2.11 Is it possible for a time-varying system $\dot{x}(t) = A(t)x(t)$ to have all its eigenvalues in the right half plane and also be stable in the sense that $\|\Phi(t, t_0)\| \to 0$ as $t \to \infty$?

Hand in problems - to be handed in at exercise session

Handin 2.1 Based on the conclusions in Exercise 2.8, derive expressible formulas of the transition matrix $\Phi(t, t_0)$ for the time-varying matrix $A(t)$ that satisfies: (a) the third condition ($A(t) = \sum_{i=1}^{n} \alpha_i(t)B_i$); (b) the fourth condition ($A(t) = TD(t)T^{-1}$), respectively.

Note: the formula of the transition matrix for $A(t)$ that satisfies the first and second conditions is discussed in the lecture slides.

Handin 2.2 Compute $\Phi(t, \tau)$ for

$$A(t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}.$$

Hint: Decompose the matrix $A(t)$ as $A(t) = a_1(t)A_1 + a_2(t)A_2$ where A_1 and A_2 commute.

Handin 2.3 Construct some examples for time-varying linear systems $\dot{x}(t) = A(t)x(t)$ in $\mathbb{R}^3$, such that

- The system is asymptotically stable, while the coefficient matrix $A(t)$ has some eigenvalues in the RHPL all the time;
- The system is unstable, while the coefficient matrix $A(t)$ has all eigenvalues in the LHPL all the time.

You may simulate your constructed systems in Matlab/Maple to test their stability/instability.

Hint: follow the $\mathbb{R}^2$ system example in the lecture.

Handin 2.4 = Rugh 20.12