Session 0

Math background

Reading Assignment

Get the book.
If you cannot get a hard copy, here is the link for an e-copy (Link expired by Dec. 2019):
`https://www.dropbox.com/s/03tsrmwmvsik5w5/Rugh_Linear%20System%20Theory.pdf?dl=0`

Exercise 0.1 Compute e^{At} for $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

Exercise 0.2 Given a positive semi-definite and symmetric matrix $A \in \mathbb{R}^{n \times n}$ where only one of its eigenvalues is 0 and all its column entries sum up to 0, there exists a matrix $U \in \mathbb{R}^{n \times n}$ satisfying $U^T U = U U^T = I$ such that

$$U^T A U = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix},$$

where λ_i for $i \in \{2, \dots, n\}$ is an eigenvalue of A. Show that $U = [\ \frac{1}{\sqrt{n}} 1_n \quad U_2\ ]$, where 1_n denotes a vector with n-entries of 1. Find U_2.

Exercise 0.3 Prove the Courant-Fisher formula for symmetric A

$$\lambda_{max}(A) = \max_{\|x\|=1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{x^T x},$$

Hint: use the decomposition $A = U \Lambda U^T$, where $U U^T = I$.

Exercise 0.4 Given a $m \times n$ matrix A, show that the spectral norm of A is given by

$$\|A\|_2 = \left(\max_{\|x\|=1} x^T A^T A x \right)^{1/2}.$$

Conclude that $\|A\|_2 = \left(\lambda_{max}(A^T A)\right)^{1/2} = \sigma_{max}(A)$ (the maximum singular value of A).

Exercise 0.5 Suppose $A(t)$ and $B(t)$ are matrices with entries which are differential functions of t. Show that the following holds

$$\frac{d}{dt}[A(t)B(t)] = \left(\frac{d}{dt} A(t) \right) B(t) + A(t) \left(\frac{d}{dt} B(t) \right).$$

Definition: The derivative of a matrix is defined entry-by-entry.

Exercise 0.6 Evaluate the derivative of the inverse of matrix $A(t)$, i.e. find $\frac{d}{dt}A(t)^{-1}$. Hint: use the results above.

Exercise 0.7 Show that if A is symmetric with $0 \prec aI \preceq A \preceq bI$, then

$$0 \prec b^{-1}I \preceq A^{-1} \preceq a^{-1}I$$

Exercise 0.8 Let $\|A\|_F$ be the Frobenius norm of $A \in \mathbb{R}^{n\times n}$. Show that $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2$, where σ_i for $i \in \{1, \ldots, r\}$ is a singular value of A and r is the rank of A.

Exercise 0.9 Show that $\|AB\|_F \leq \|A\|_F\|B\|_F$, i.e. that the Frobenius-norm is submultiplicative.

Exercise 0.10 Show that $q \in null(A^T) \iff q \perp range(A)$.

Hand in problems - to be handed in at exercise session

Handin 0.1. Use Matlab and/or Maple to calculate characteristic polynomial, eigenvalues, eigenvectors and e^{At} both numerically and symbolically for $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

Handin 0.2: Prove or disprove the following statements:
Given two real symmetric matrices $A, B \in \mathbb{R}^{n\times n}$:

1. Suppose A and B are both positive definite. Then AB are symmetric and positive definite.
2. Suppose A and B are both positive definite. Then all eigenvalues of AB are real and positive.
3. Suppose $0 \prec A \preceq B$. Then $0 \prec A^{\frac{1}{2}} \preceq B^{\frac{1}{2}}$.
4. Suppose $0 \prec A \preceq B$. Then $0 \prec A^2 \preceq B^2$.

Handin 0.3: Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with the input

$$u(t) = u_0\delta(t) + u_1\delta^{(1)}(t) + \ldots + u_r\delta^{(r)}(t).$$

where the u_k are constants, $\delta(t)$ is the dirac impulse function and $\delta^{(i)}(t)$ denotes its i-th derivative. Let $\theta(t)$ denote the step function $\theta(t) = 1_{t\geq 0}$, and $\theta'(t) = \delta(t)$. Show that there is a solution of the form

$$x(t) = e^{At}v_0\theta(t) + v_1\delta(t) + \ldots + v_r\delta^{(r-1)}(t)$$

and determine the vectors v_k.