Model Uncertainty and Robust Control Design

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1. Introduction
2. Classical Robustness Theory
3. State Space Design
4. Limitations on Control System Performance
5. $\mathcal{H}_\infty$ Loop Shaping
6. Conclusions

1. Introduction

- Ways of coping with uncertainty have been a key motivation for use of feedback
- Without uncertainty there is no need for feedback
- A brief history
  - Black, Bode and Nyquist
  - Blacks ideal loop transfer function
  - Horowitz and QFT
  - State space theory
  - $\mathcal{H}_\infty$, Zames, Glover, Doyle, ...
- How to cope with uncertainty
  - Live with it: Robust control!
  - Reduce it: Adaptive control!

The Feedback Amplifier

The repeater problem

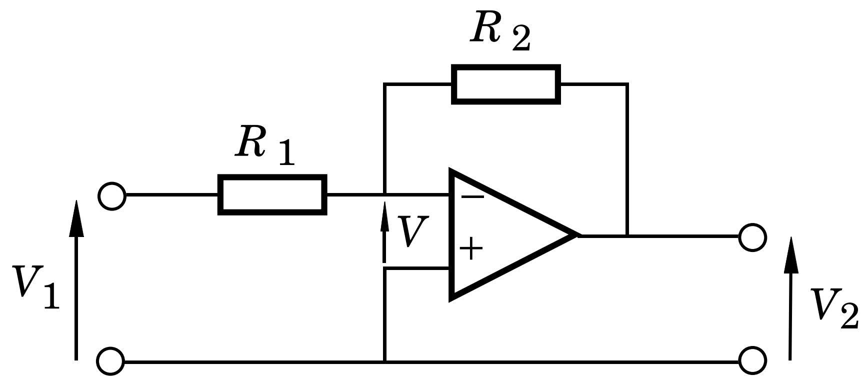
Blacks invention 1927

Nyquist 1932

Blacks paper 1934

Bode 1940

Bodes book 1945



$$\frac{V_2}{V_1} = -\frac{R_2}{R_1}\frac{1}{1+\frac{1}{A}\left(1+\frac{R_2}{R_1}\right)}$$

Mervin Kelly on Black at the presentation of the IEEE Lamme Medal 1957

It easily ranks coordinate with De Forest's invention of the audion as one of the two inventions of broadest scope and significance in electronics and communications of the past 50 years. It is no exaggeration to say that without Black's invention (of the feedback amplifier), the present long-distance telephone and television networks which cover our entire country and the transoceanic telephone cables would not exist. The application of Black's principle of negative feedback has not been limited to telecommunications. Many of the industrial and military amplifiers would not be possible except for its use.

Theoretical Insight

- Nyquist 1932
- Bode 1940
- Important ideas

  Nyquist curve

  Bode diagram

  Bodes relations

  Bodes integrals

  Bodes ideal loop transfer function
- Horowitz 1963 +

  Templates

  Qualitative Feedback Theory (QFT)

State Space Theory

- Many useful concepts

  State

  Observability, reachability

  Kalman filters and separation

- Uncertainty treated as parameter errors or additive disturbances
- Difficult to deal with unmodeled dynamics
- Multi-variable systems

  Singular values are what matters for robustness!

- H_∞ theory

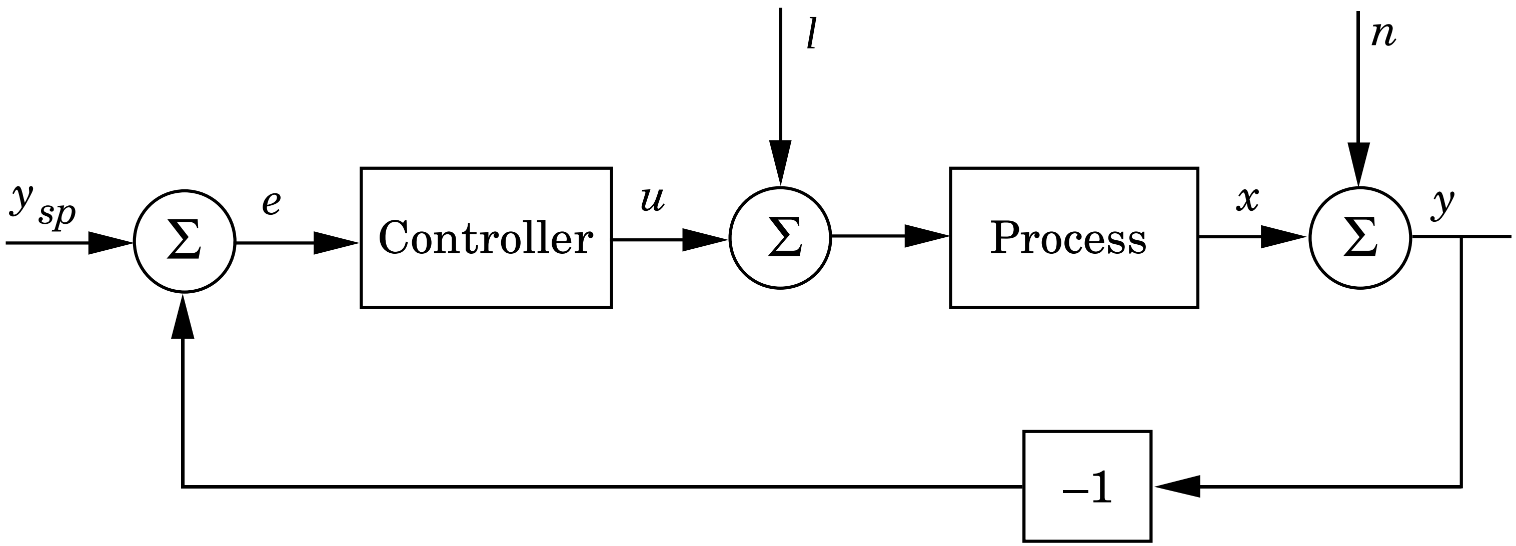
  Brought uncertainty into the picture again!

  Structured uncertainty and μ

2. Classical Robustness Theory

- The simple feedback loop
- The fundamental transfer functions
- Small model errors
- Large model errors
- The sensitivity functions
- Bode's relations and integrals
- The delay margin
- Bode's ideal loop transfer function
- Fractional systems
- Systems with two degrees of freedom
- Summary

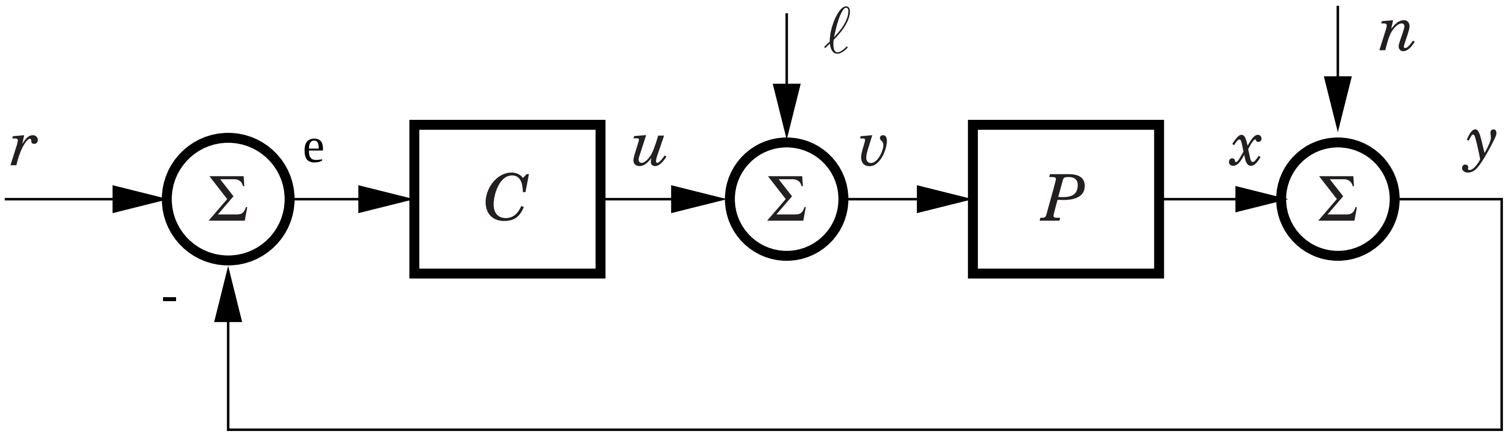
The Standard Problem



Desirable properties

- Reduce effects of load disturbances
- Make sure that measurement noise has small effect
- Low sensitivity to process model
- Follow command signals well

The Basic Feedback Loop



$$G_{xr} = \frac{PC}{1+PC} \qquad G_{x\ell} = \frac{P}{1+PC} \qquad G_{xn} = -G_{xr}$$

$$G_{yr} = G_{xr} \qquad G_{y\ell} = G_{x\ell} \qquad G_{yn} = \frac{1}{1+PC}$$

$$G_{er} = 1 - G_{xr} = G_{yn} \qquad G_{e\ell} = -G_{x\ell} \qquad G_{en} = -G_{yn}$$

$$G_{ur} = \frac{C}{1+PC} \qquad G_{u\ell} = -G_{xr} \qquad G_{un} = -G_{ur}$$

The Basic Transfer Functions

Three inputs $y_{sp} = r$, l and n, four interesting signals x, y, e and u gives 12 transfer functions

$$G_{xr} = \frac{PC}{1+PC} \qquad G_{x\ell} = \frac{P}{1+PC} \qquad G_{xn} = -G_{xr}$$

$$G_{yr} = G_{xr} \qquad G_{y\ell} = G_{x\ell} \qquad G_{yn} = \frac{1}{1+PC}$$

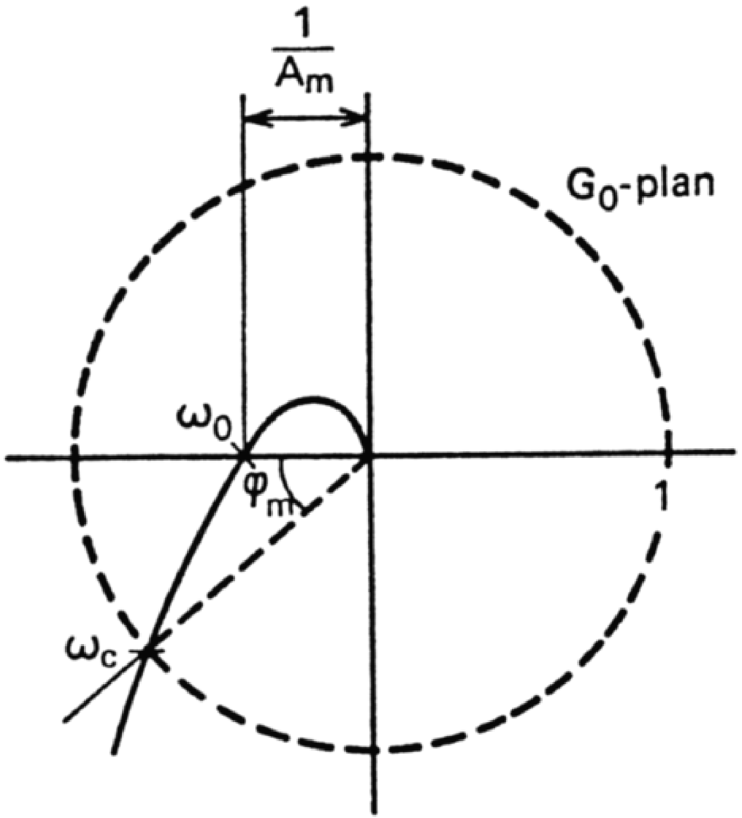
$$G_{er} = 1 - G_{xr} = G_{yn} \qquad G_{e\ell} = -G_{x\ell} \qquad G_{en} = -G_{yn}$$

$$G_{ur} = \frac{C}{1+PC} \qquad G_{u\ell} = -G_{xr} \qquad G_{un} = -G_{ur}$$

Sufficient to study “The Gang of Four” G_{xr}, G_{xl}, G_{yn} and G_{ur}.

Nyquist's Stability Criterion

$$\frac{1}{2\pi}\Delta \arg_{\Gamma}(1 + L(s)) = -\mathcal{P}_{rhp}(L)$$



- Stability margins
- Gain margin
- Phase margin
- Maximum sensitivity

Small Process Variations

The closed loop transfer function is

$$T = \frac{PC}{1 + PC}$$

How much does T change when the process changes?

$$\frac{dT}{T} = \frac{dP}{P} - \frac{CdP}{1 + PC} = \frac{1}{1 + PC}\frac{dP}{P} = S\frac{dP}{P}$$

The sensitivity function S tells how the closed loop properties are influenced by small variations in the process.

Large Variations

How much can the process P change without making the closed loop system unstable?

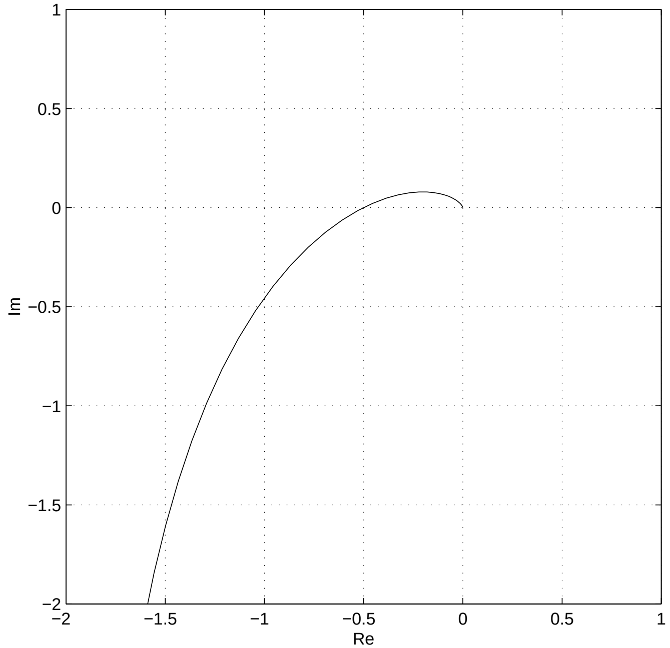
$$|C\Delta P| < |1 + PC|$$

Hence

$$|\frac{\Delta P}{P}| < \left|\frac{1+PC}{PC}\right| = \left|\frac{1}{T}\right|$$

Conservative bound

$$|\frac{\Delta P}{P}| \le \frac{1}{M_t}$$



Large variations when the closed loop system has large gain compared to the closed loop system. Compare with Black!

Small Gain Interpretation

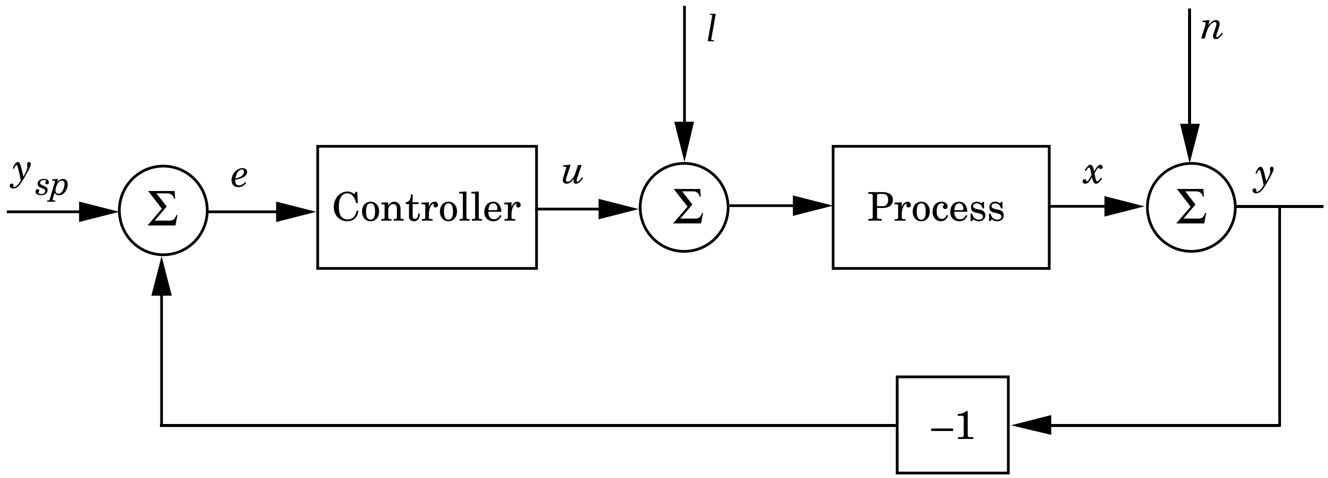
$$|\frac{\Delta P}{P}| < \left|\frac{1+PC}{PC}\right| = \left|\frac{1}{T}\right|$$

Hence

$$|T\frac{\Delta P}{P}| < 1$$

Draw pictures on the board!

The Sensitivity Function



Output under open loop control: $Y_{ol}(s) = N(s) + PL$

Output under closed loop control: $Y_{cl} = \frac{1}{1+PC}(N + PL)$

Hence

$$\frac{Y_{cl}}{Y_{ol}} = \frac{1}{1 + PC} = S$$

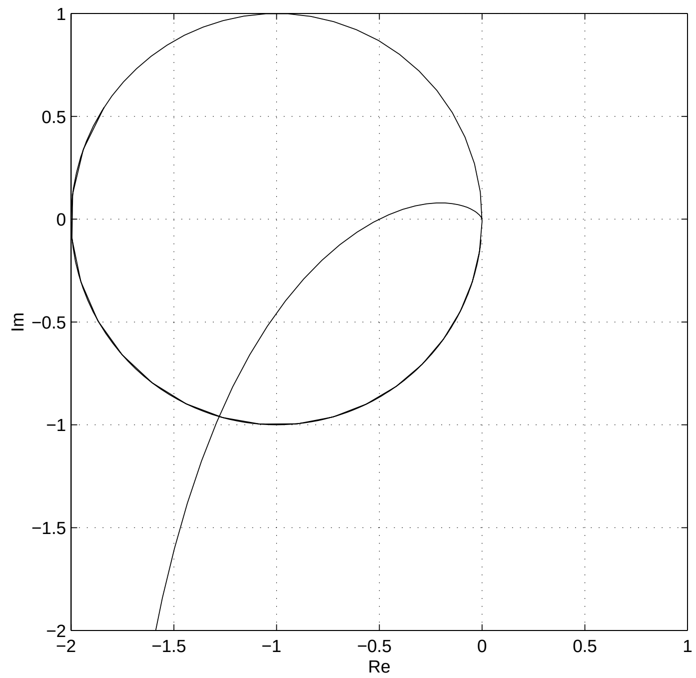
The sensitivity function tells how the variations in the output are influenced by feedback!

The Sensitivity Function

Variations are reduced for frequencies outside the circle! They are amplified for frequencies inside the circle.
Fundamental limitations are given by Bodes integral

$$\int_0^\infty \log |S(i\omega)| d\omega = \pi \sum p_k$$

$$\int_0^\infty \log |T(1/i\omega)| d\omega = \pi \sum \frac{1}{z_k}$$



The Sensitivity Function and Bode's Integrals

$$S = \frac{1}{1+L}, \qquad M_s = \max |S(i\omega)|$$

The value $1/M_s$ minimum distance from the Nyquist curve of the loop transfer function $L(i\omega)$ to the critical point -1.

$$S = \frac{\partial \log T}{\partial \log P} = \frac{Y_{cl}}{Y_{ol}}$$

Bode's integrals: $sL(s)$ goes to zero as $s \to \infty$

$$\int_0^\infty \log |S(i\omega)| d\omega = \int_0^\infty \log |\frac{1}{1+L(i\omega)}| d\omega = \pi \sum p_i$$

$$\int_0^\infty \log |T(1/i\omega)| d\omega = \int_0^\infty \log |\frac{L(1/i\omega)}{1+L(1/i\omega)}| d\omega = \pi \sum \frac{1}{z_i}$$

The water bed effect.

Bode's Relations

G rational no RHP poles and zeros

$$\log G(i\omega) = A(\omega) + i\Phi(\omega), \qquad u = \log \frac{\omega}{\omega_0},$$

$$\omega = \omega_0 e^u, \qquad a(u) = A(\omega_0 e^u), \qquad \phi(u) = \Phi(\omega_0 e^u)$$

$$\begin{aligned} A(\omega_0) - A(\infty) &= -\frac{2}{\pi}\int_0^\infty \frac{v\Phi(v) - \omega_0\Phi(\omega_0)}{v^2 - \omega_0^2} dv \\ &= -\frac{1}{\omega_0 \pi}\int_{-\infty}^\infty \frac{d(e^u \phi(u))}{du} \log\coth|\frac{u}{2}| du \\ \Phi(\omega_0) &= \frac{2\omega_0}{\pi}\int_0^\infty \frac{A(v) - A(\omega_0)}{v^2 - \omega_0^2} dv = \frac{1}{\pi}\int_{-\infty}^\infty \frac{da(u)}{du} \log\coth|\frac{u}{2}| du \end{aligned}$$

Approximately: $\Phi(\omega) \approx \frac{2}{\pi}\frac{da(u)}{du}$.

Bode's Relations

$$A(\omega_0) - A(\infty) = -\frac{1}{\omega_0 \pi} \int_{-\infty}^{\infty} \frac{d(e^u \phi(u))}{du} \log \coth |\frac{u}{2}| du$$

$$\Phi(\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{da(u)}{du} \log \coth |\frac{u}{2}| du$$

an approximate version is that

$$\Phi(\omega) \approx \frac{2}{\pi} \frac{da(u)}{du}.$$

Can you beat Bode's relations by nonlinear compensators?
Yes, the Clegg integrator is one example!

The Delay Margin (Semi-classic)

The traditional gain and phase margins are based on one intersection with the negative real axis or one intersection of the unit circle.

- What happens when there are several intersections?
- When does that happens?
  - Dead time compensation
  - Systems with resonances
- Traditional robustness measures can be very misleading for such systems!
- Delay margin!
- The smallest time delay that can be introduced without driving the system unstable.

Example - A Simple Predictor

The regular PID controller

$$u = ke + \frac{1}{k_i}\int^t e(s)ds - k_d \frac{dy_f}{dt}$$

The PPI (Predictive PI) controller

$$u = ke + \frac{1}{k_i}\int^t e(s)ds - k_p \int_{t-L}^{t} u(s)ds$$

Predict using past values of the control signal instead of the derivative of the output!

The Smith Predictor

Process transfer function $G_p = Ge^{-sL}$, controller transfer function G_c.

Equivalent controller

$$\tilde{G}_c = G_c \frac{1}{1 + G_m G_c (1 - e^{-sL})}$$

Closed loop transfer function

$$G_{cl} = \frac{GG_c}{1 + GG_c} e^{-sL}$$

Sensitivity function (Note always $M_s \leq 2$)

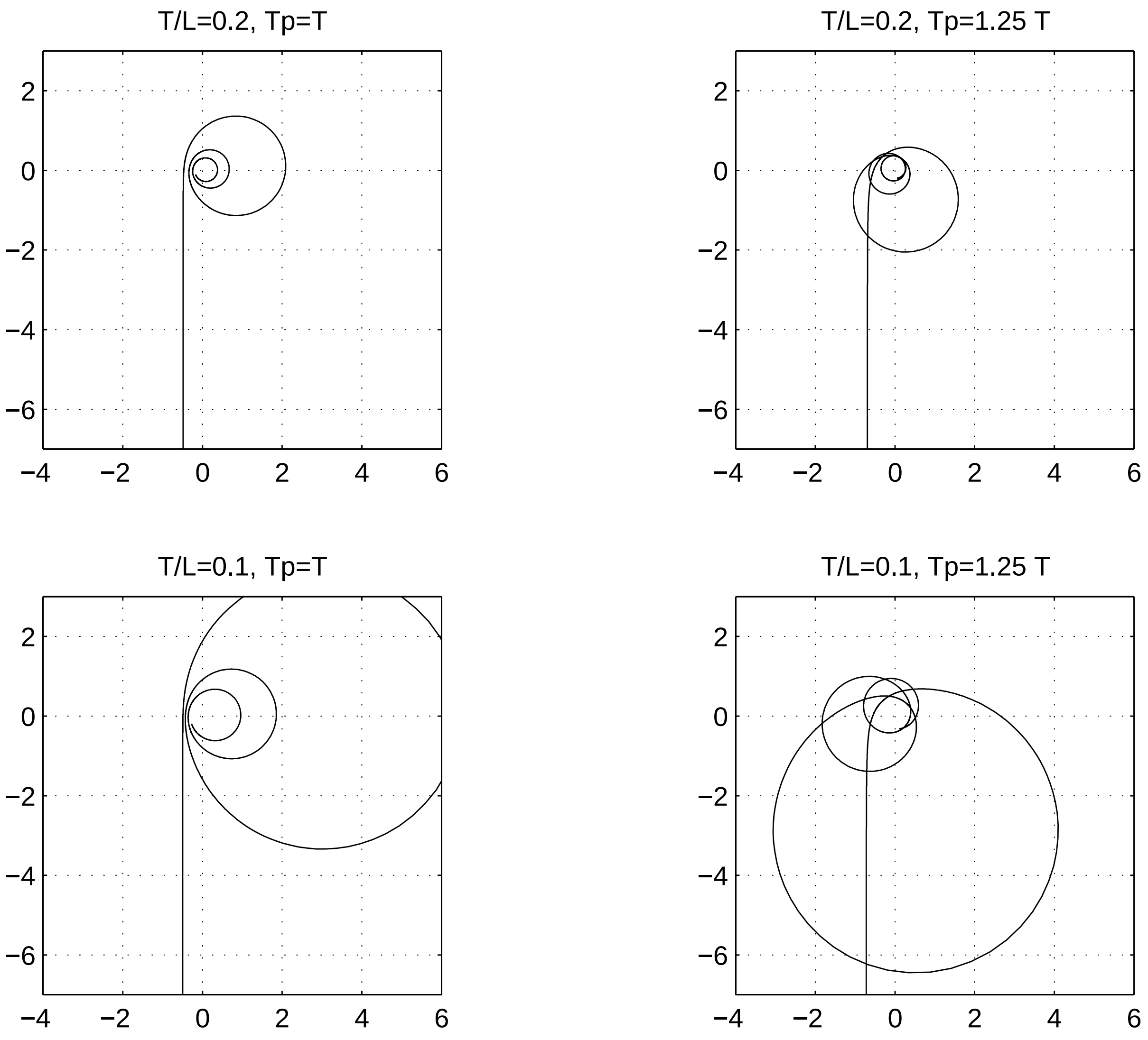
$$S = 1 - \frac{GG_c}{1 + GG_c} e^{-sL} = 1 - G_{cl}$$

The Danger of Being Greedy

System has time delay L. Desired response time T? Sensitivity function

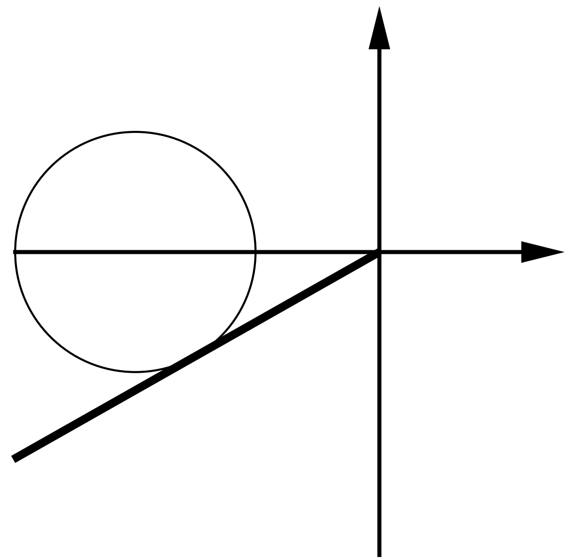
$$S = 1 - \frac{1}{1+sT}e^{-sL}$$

We have $M_s \leq 2$ which gives a tolerable robustness.



Bodes Ideal Loop Transfer Function

The repeater problem. Large gain variations in tube amplifiers What should a transfer function look like to be independent of gain?



$$L(s) = \left(\frac{s}{\omega_{gc}}\right)^n$$

Phase margin invariant with loop gain
$n = -1.5$ gives $\varphi_m = 45^\circ$

Horowitz extended Bodes ideas to deal with arbitrary plant variations not just gain variations in the QFT method.

Fractional Systems

Consider the process

$$P(s) = \frac{k}{s(s+1)}$$

Find a controller that gives a phase margin of 45°. Bodes ideal loop transfer function is

$$L(s) = \frac{1}{s\sqrt{s}}$$

Since $L = PC$ we find that the controller transfer function is

$$C(s) = \frac{s+1}{\sqrt{s}} = \sqrt{s} + \frac{1}{\sqrt{s}}$$

A fractional transfer function!

Implementation of a Fractional Controller

Approximate by rational transfer function. There are many ways to do this. One possibility is:

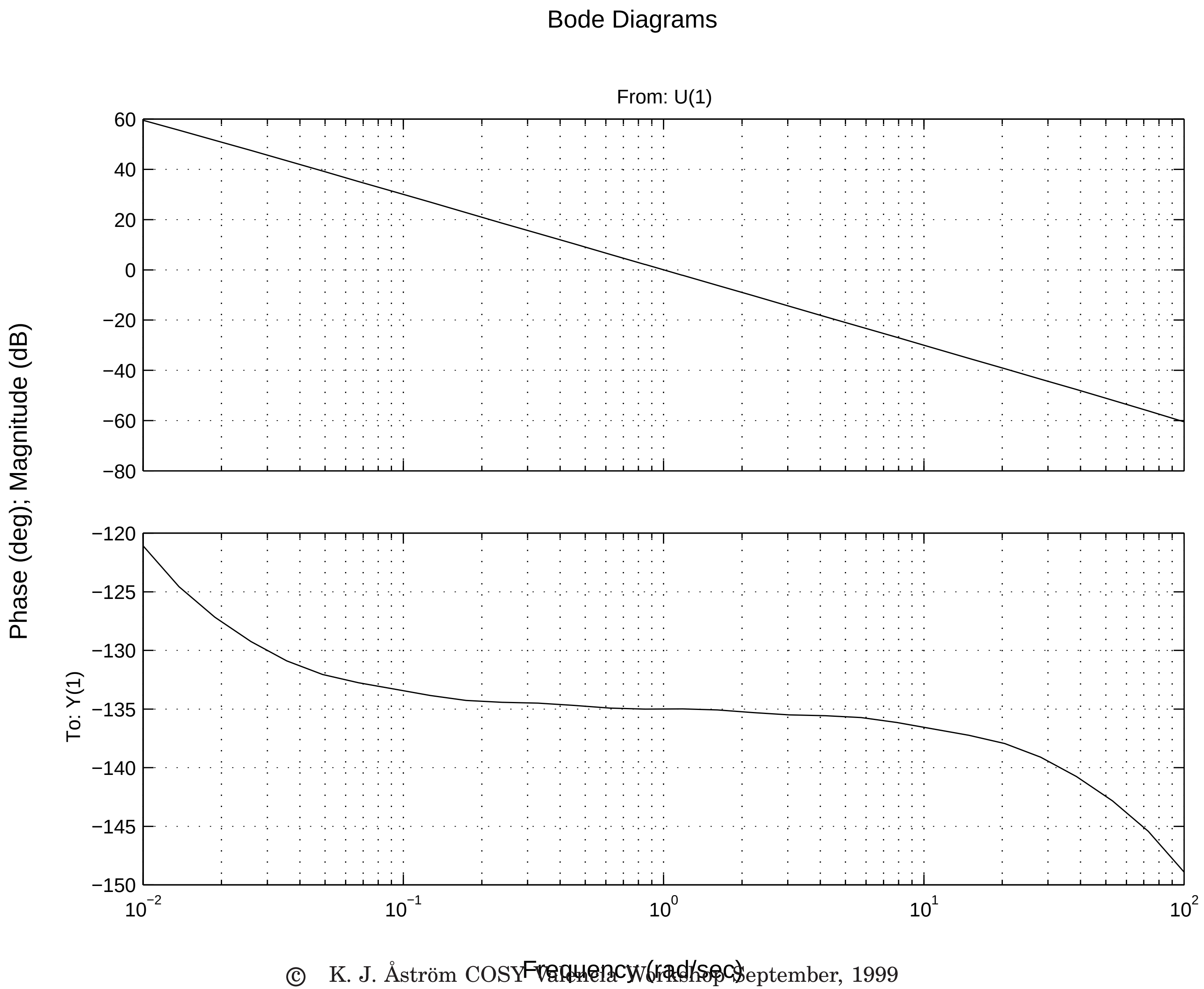
$$\hat{C}(s) = \frac{k(s+1/64)(s+1/16)(s+1/4)(s+1)^2(s+4)(s+16)(s+64)}{(s+1/128)(s+1/32)(s+1/8)(s+1/2)(s+2)(s+8)(s+32)(s+128)}$$

Gain k is chosen to equal the gain of $\sqrt{s} + 1/\sqrt{s}$ for $s = i$. Notice that the controller is composed of sections of equal length having slopes 0, +1 and -1 in the Bode diagram.

How large gain variations can the system handle?

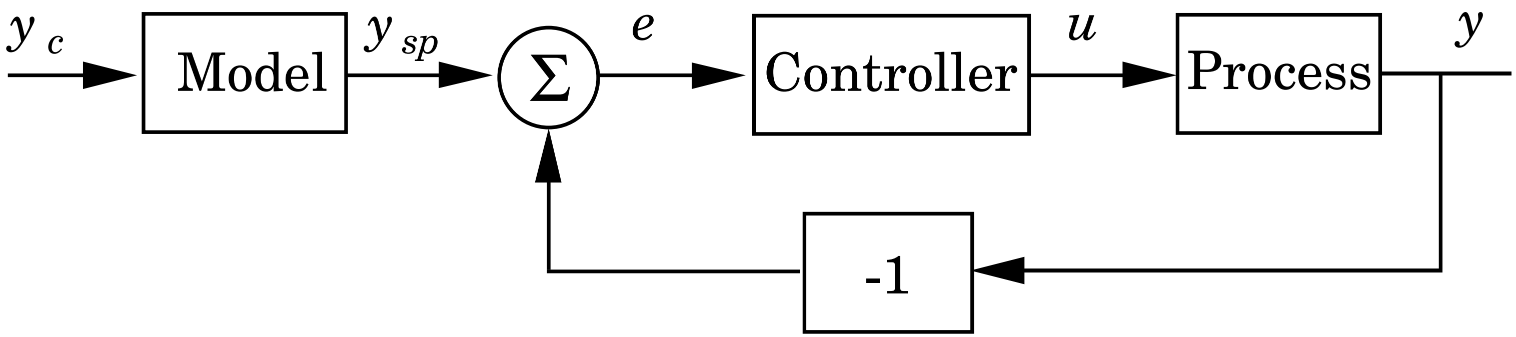
Notice that improved robustness requires higher order compensator!

Bode Diagram of Loop Transfer Function

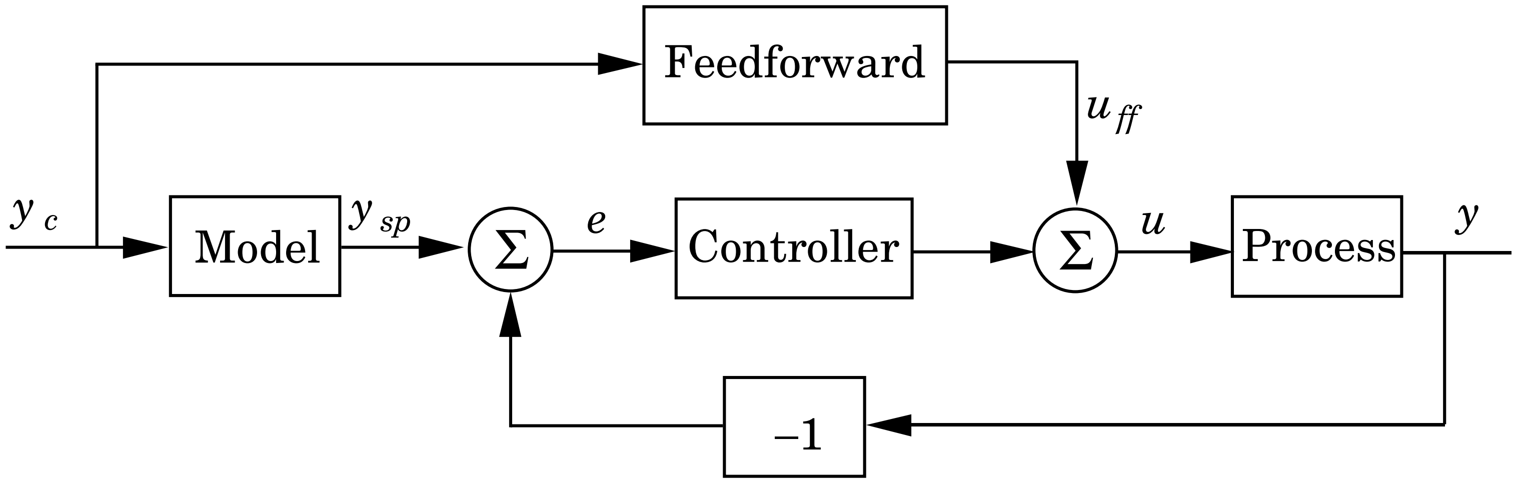


Two Degree of Freedom Systems (2DOF)

Separate regulation from command signal following.



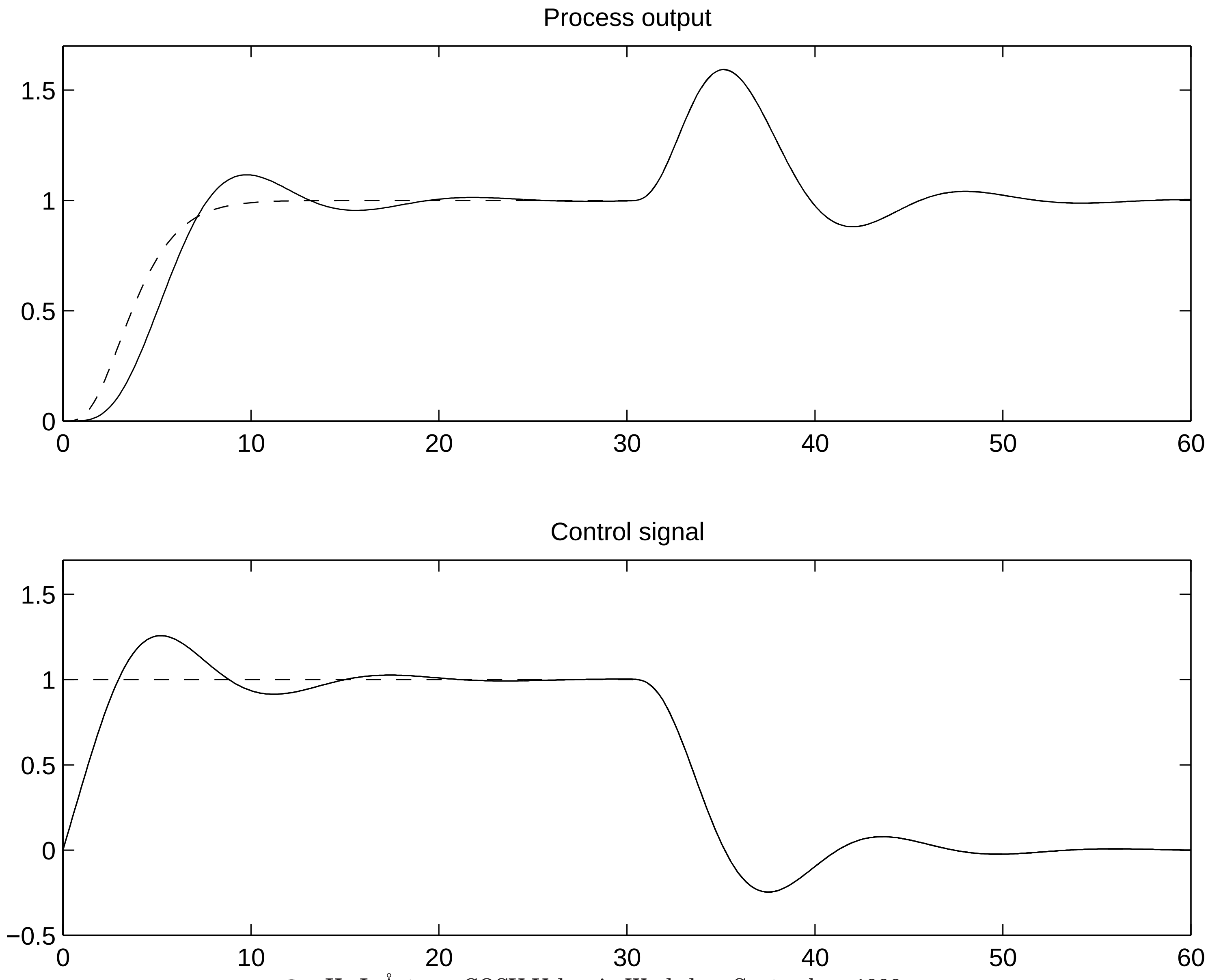
A better system



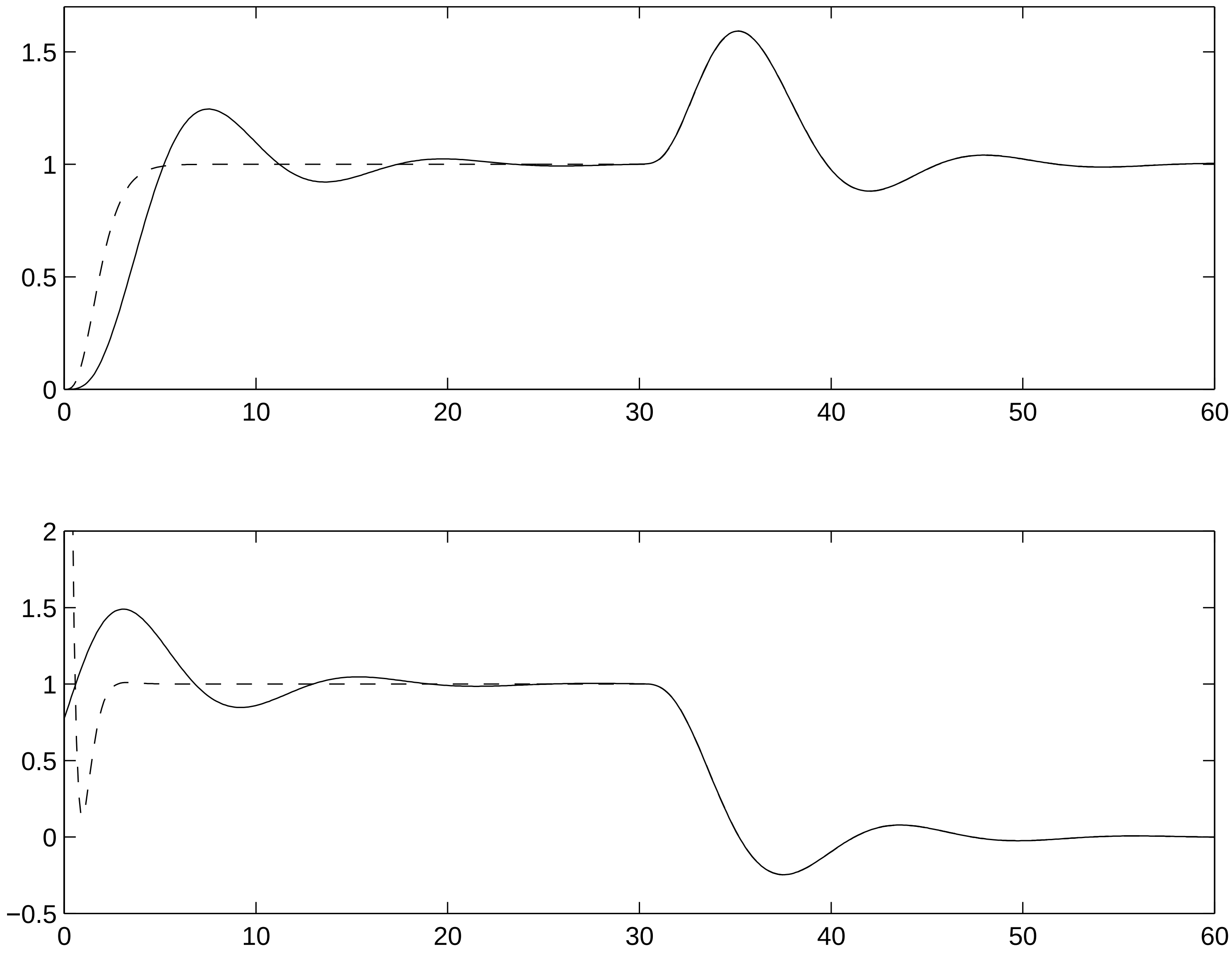
Two Degree of Freedom Systems

Horowitz 1963: "Some structures have been presented as fundamentally different from the others. It has been suggested that they have virtues not possessed by others, and have been given special names ... all 2DOF configurations have basically the same properties and potentials ...."

An Example


Process output
1.5
1
0.5
0
0
10
20
30
40
50
60
Control signal
1.5
1
0.5
0
−0.5
0
10
20
30
40
50
60

Fast Response at a Cost $u(0) = 16$



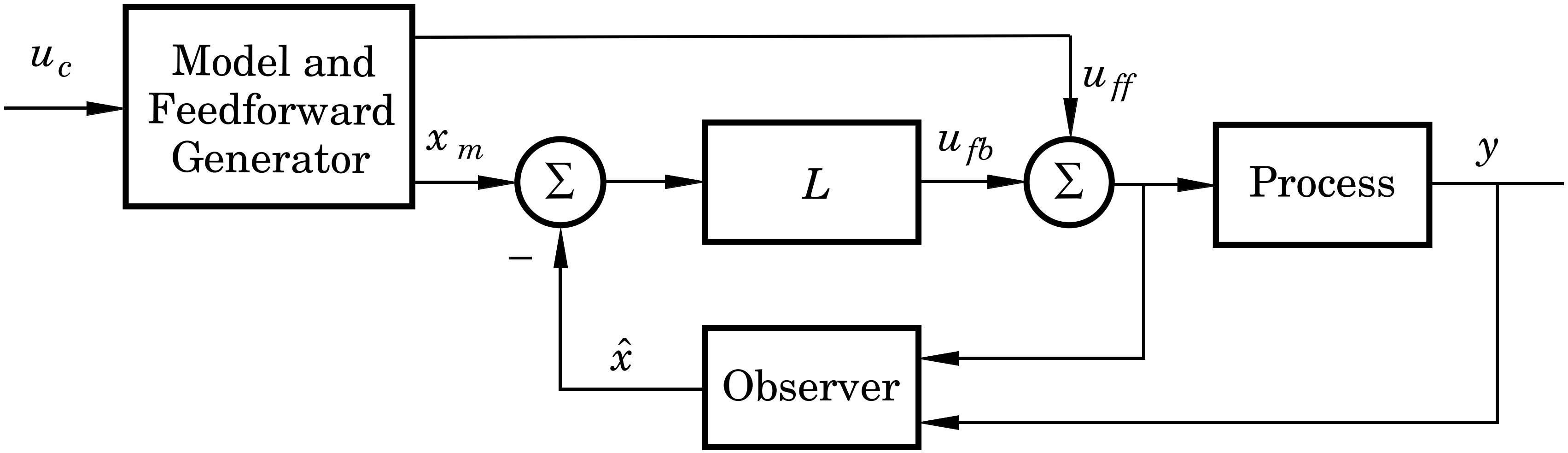
Summary

- Process uncertainty was an essential ingredient i servomechanism theory, note perturbations restricted
- Uncertainty was in fact the key reason for introducing feedback
- Uncertainty could easily be expressed in terms of transfer function
- Many loop shaping design methods took uncertainty explicitly into account
- The role of two degree of freedom configurations was well understood
- Horowitz Qualitative Feedback Theory (QFT) is the natural conclusion of the classical results

3. State Feedback

- A new paradigm
- New controller structures, new insights and efficient design algorithms
- Difficult to capture uncertainty
  - Parameter uncertainty
  - View uncertainty as a disturbance
  - Easy to make mistakes
  - You must always check robustness after a design!
  - LQG-LTR was one attempt to recover robustness

The controller structure



Assessment

The state space approach has given us many nice design methods but it is very easy to formulate innocent design problems that give extremely non-robust systems.

- The only requirements are observability and reachability
- Robustness is not part of the problem formulation
- Robustness must always be checked afterwards
- Fundamental insights required

An Example

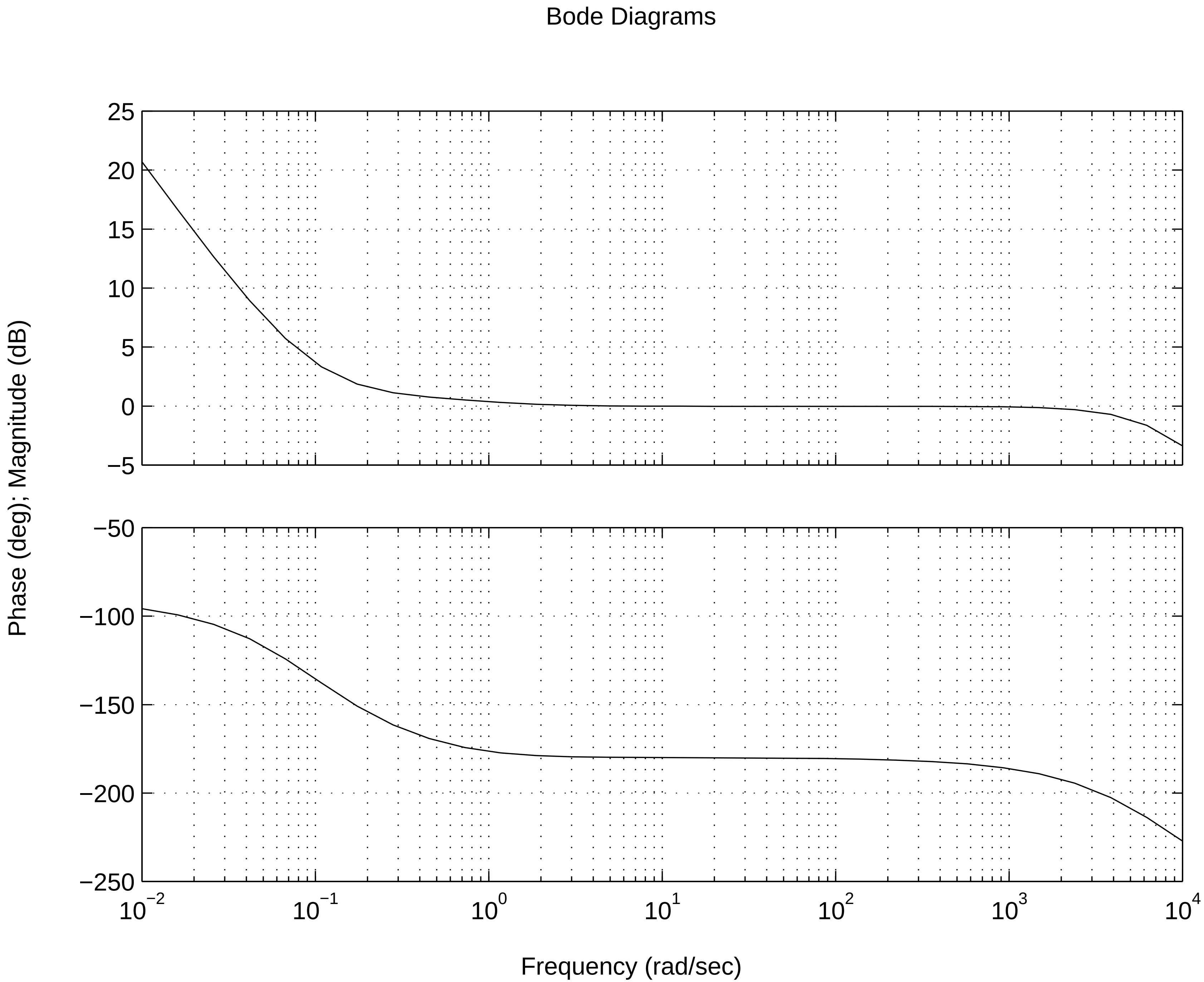
Consider the system

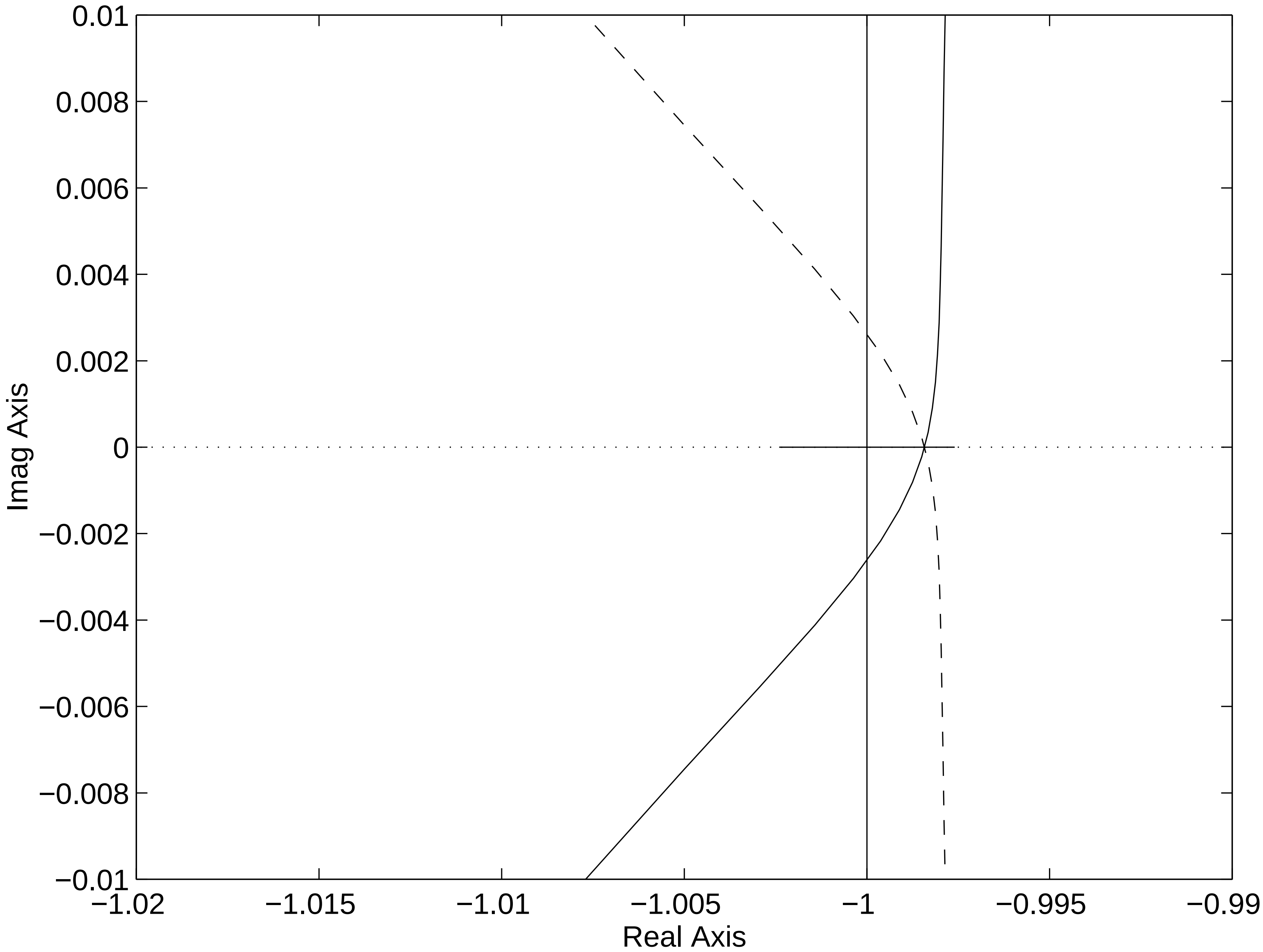
$$\frac{dx}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} a \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

with $a = -10$. Design a state feedback that gives the closed loop characteristic polynomial

$$(s + \alpha\omega_0)(s^2 + \omega_0 s + \omega_0^2)$$

with $\alpha = 1$ and $\omega_0 = 10$. Butterworth configuration!


Bode Diagrams
Phase (deg); Magnitude (dB)
25
20
15
10
5
0
−5
−50
−100
−150
−200
−250
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/sec)


0.01
0.008
0.006
0.004
0.002
0
−0.002
−0.004
−0.006
−0.008
−0.01
Imag Axis
−1.02
−1.015
−1.01
−1.005
−1
−0.995
−0.99
Real Axis

Properties of the System

Loop transfer function

$$
\begin{aligned}
L(s) &= \frac{(as+1)(s_0 s + s_1)}{s(s+1)(s+r)} \\
&= -9255\frac{(s-0.1)(s+1.0805)}{s(s+1)(s+9274)}
\end{aligned}
$$

Stability margins

$$
\begin{aligned}
\omega_{gc} &= 6.58 \\
\varphi_m &= 6.6^\circ \\
M_s &= 678 \\
M_t &= 677
\end{aligned}
$$

A Very Bad Closed Loop System!
Why?

4. Fundamental Limitations

The reason why we get strange results is that robustness was not part of the formulation and that we had formulated an design problem that does not result in a robust system. How can we understand this?

- There are fundamental limitations on control system design
- We must know how to express these so that we are not formulation ridiculous problems
- Limitations are given by RHP poles and zeros and time delays

Introduction

Goal: Capture some essential difficulties of a design problem in a simple way before attempting detailed design calculations

- Find appropriate concepts
  - Controllability index L/T
  - Bode: RHP poles and zeros
  - Reachability
  - Observability
- Inspiration
  - Bodes 1945 book
  - Freudenberg and Looze 1988
  - Seron, Braslavsky and Goodwin 1997
  - Stein's Bode Lecture

The Crossover Frequency Inequality

Process $P(s) = P_{mp}(s)P_{nmp}(s)$, controller $C(s)$

$$\arg L(i\omega) = \arg P_{nmp}(i\omega) + \arg P_{mp}(i\omega) + \arg C(i\omega) \geq \pi + \varphi_m$$

Bode's ideal cut off characteristic

$$\arg P_{mp}(i\omega) + \arg C(i\omega) + \arg \Delta P_{nmp}(i\omega_{gc}) = n\frac{\pi}{2}$$

Crossover Frequency Inequality

$$\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \varphi_m - n_{gc}\frac{\pi}{2} - \arg \Delta P_{nmp}(i\omega_{gc})$$

Simple Rule of Thumb

$$\arg P_{nmp}(i\omega_{gc}) \geq -\frac{\pi}{2}$$

Non-minimum phase components can have a phase-lag of at most 45° at the gain cross over frequency!

System with RHP Zero

Assume

$$G_{nmp}(s) = \frac{a-s}{a+s}$$

Hence

$$\arg G_{nmp}(i\omega_{gc}) = -2\arctan\frac{\omega}{a}$$

Cross over frequency inequality

$$n_{gc}\frac{\pi}{2} - 2\arctan\frac{\omega_{gc}}{a} \geq -\pi + \varphi_m$$

Hence

$$\frac{\omega_{gc}}{a} \leq \tan(\frac{\pi}{2} - \frac{\varphi_m}{2} + n_{gc}\frac{\pi}{4})$$

Example $\varphi_m = \pi/4$ and $n_{gc} = -1/2$ gives $\omega_{gc} < a$

Return to the Example

Process dynamics

$$G(s) = \frac{-10s + 1}{s(s + 1)}$$

Design a closed loop system that has poles in $s = -10$ and $s = -5 \pm i\sqrt{75}$!

The process has a right half plane zero at $s = 0.1$ and we have required closed loop poles in a Butterworth configuration with radius 10 i.e a cross over frequency of approximately $\omega_{gc} \approx 10$.

Remember limitations! A right half plane zero at $s = \alpha$ limits the crossover frequency to $\omega_{gc} < \alpha$. In this case we had $\omega_0 = 10$ which is a strong violation of $\omega_{gc} < 0.1$!!!

System with Time Delay

$$G_{nmp}(s) = e^{-sL}$$

Cross over frequency inequality

$$\omega_{gc} L \leq \pi - \varphi_m + n_{gc} \frac{\pi}{2}$$

Example Assume that we want a phase margin $\varphi_m = \pi/4$ and a slope at the crossover frequency of $n_{gc} = -1/2$, then

$$\omega_{gc} L \leq \frac{\pi}{2}$$

System with RHP Pole

$$G_{nmp}(s) = \frac{s+b}{s-b}$$

Hence

$$\arg G_{nmp}(i\omega_{gc}) = -2\arctan\frac{b}{\omega}$$

Cross over frequency inequality

$$n_{gc}\frac{\pi}{2} - 2\arctan\frac{b}{\omega_{gc}} \geq -\pi + \varphi_m$$

Hence

$$\omega_{gc} \geq \frac{b}{\tan(\pi/2 - \varphi_m/2 + n_{gc}\pi/4)}$$

Example $\varphi_m = \pi/4$ and $n_{gc} = -1/2$ gives $\omega_{gc} \geq b$

System with RHP Pole and Zero

$$G_{nmp}(s) = \frac{(a-s)(s+b)}{(a+s)(s-b)}$$

For $a > b$ we have

$$\arg G_{nmp}(i\omega) = -2 \arctan \frac{\omega_{gc}/a + b/\omega_{gc}}{1 - b/a}$$

Cross over frequency inequality

$$\frac{\omega_{gc}}{a} + \frac{b}{\omega_{gc}} \le (1 - \frac{b}{a}) \tan\Big(\frac{\pi}{2} - \frac{\varphi_m}{2} + n_{gc}\frac{\pi}{4}\Big)$$

Simple calculations give

$$\frac{a}{b} \ge 1 + \frac{2 + 2\sqrt{1+\alpha^2}}{\alpha^2}$$

The Cross over Frequency Inequality

$$\frac{a}{b} \geq 1 + \frac{2 + 2\sqrt{1+\alpha^2}}{\alpha^2}$$

Assume a phase margin $\varphi_m = \pi/4$ and a slope at the crossover frequency of $n_{gc} = -1/2$, then

$$a \geq 5.83b$$

Phase margin

$$\varphi_m \leq \pi + n_{gc}\frac{\pi}{2} - 2\arctan\frac{2\sqrt{ab}}{a-b}$$

a/b	2	5	10	20
φ_m	-6.0	38.6	64.8	84.6

Example - The X-29

Non-minimum phase part of the transfer function

$$G_{nmp}(s) = \frac{s - 26}{s - 6}$$

hence $a/b = 4.33$ with $n_{gc} = -1/2$ we get

$$\varphi_m = 32.4$$

A phase margin of 45° cannot be achieved.

With the slope $n_{gc} = -0.36$ it is possible to get $\varphi_m = 45^\circ$.

Adding Model Uncertainty

The crossover frequency inequality

$$\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \varphi_m - n_{gc}\frac{\pi}{2} - \arg \Delta P_{nmp}(i\omega_{gc})$$

This equation shows the effect of model uncertainty. Phase uncertainty at the crossover frequency is critical.

Assume for example that there is an uncertainty in the time delay ΔT, then the phase uncertainty contributes with 45° at the frequency $\omega = \pi/(4\Delta T)$.

Other Criteria

There are several alternatives to the phase margin φ_m, for example

$$M_s = \max_\omega |S(i\omega)|$$

$$M_t = \max_\omega |T(i\omega)|$$

Combined sensitivity

$$M_{sp} = \max_\omega (|T(i\omega)| + |S(i\omega)|)$$

$\mathcal{H}_\infty$ norm

$$M = \max_\omega \frac{\sqrt{(1+|C|^2)(1+|P|^2)}}{|1+PC|}$$

These criteria gives essentially the same results but the numerical values are changed.

Summary of Limitations - Part 1

- A RHP zero z

$$\frac{\omega_{gc}}{z} \leq \begin{cases} 0.5 & \text{for } M_s,\ M_t < 2 \\ 0.2 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A time delay T

$$\omega_{gc} T \leq \begin{cases} 0.7 & \text{for } M_s,\ M_t < 2 \\ 0.37 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A RHP pole p

$$\frac{p}{\omega_{gc}} \geq \begin{cases} 2 & \text{for } M_s,\ M_t < 2 \\ 5 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

Summary of Limitations - Part 2

- A RHP pole-zero pair with $z > p$

$$\frac{z}{p} \geq \begin{cases} 6.5 & \text{for } M_s,\ M_t < 2 \\ 14.4 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A RHP pole-zero pair with $z < p$

$$\frac{p}{z} \geq \begin{cases} 6.5 & \text{for } M_s,\ M_t < 2 \\ 14.4 & \text{for } M_s,\ M_t < 1.4 \end{cases}$$

- A RHP pole p and a time delay T

$$pT \leq \begin{cases} 0.16 & \text{for } M_s,\ M_t < 2 \\ 0.05 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

Summary

- For design methods that do not take robustness explicitly into account it is necessary to check robustness when the design is completed
- RHP poles and zeros and time delays give fundamental limitations
- Awareness of these limitations often make it possible to make sound design choices
- Useful to modify Bode plots so that they always show the actual phase and the minimum phase plots!

5. $\mathcal{H}_\infty$ Control

- A brief history
- Pole-zero cancellations
- The Gang of Four again
- How to compare two systems
- The Graph metric
- The Gap metric
- Vinnecombe’s metric
- Coprime factor uncertainty
- Generalized sensitivity,
- Generalized stability margins
- h_∞-loop shaping

A Brief History

- Horowitz and Shaked,1975: Superiority of transfer function of state-variable methods in linear time-invariant feedback system design
- Doyle, 1978: Guaranteed margins for LQG regulators.
- George Zames 1981: Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses, IEEE AC-26, 301-320. Brought sensitivity issues in again. Interpolation theory.
- Doyle, Glover, Khargonekar and Francis 1989: State space solutions to standard H_2 and H_∞ control problems. IEEE AC-34, 831-847. Demonstrated that the H_∞ control problem could be formulated in state space. Simplified calculations.

A Brief History

- Basar Bernard 1991, 1995: H_∞ control and related minimax design problems. A Dynamic Game approach. Showed that the results could be derived from game theory. Opened the way for nonlinear problems.
- Doyle. The μ method. Structured perturbations.
- MacFarlane and Glover, 1990. Robust controller design using normalized coprime facto plant descriptions.
- Vinnecombe 1999 Uncertainty and Feedback.

The Problem of Pole Zero Cancellation

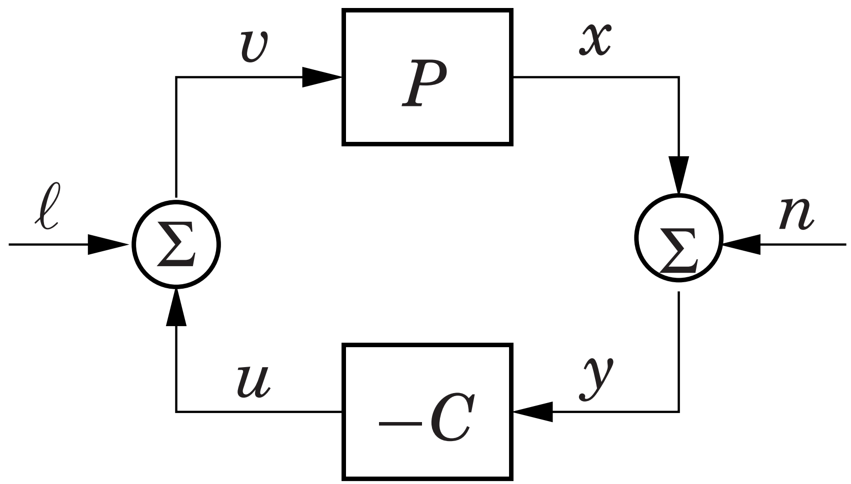
- Nyquist stability theory on uses the loop transfer function
- What happens when there are pole-zero cancellations?
- A classical dilemma in systems theory
- Module theory
- Kalmans decomposition theorem
- The “Gang of Four” formulation

An Example

Process and controller

$$C(s) = \frac{s-1}{s}$$

$$P(s) = \frac{1}{s-1}$$



Loop transfer function

$$L(s) = \frac{1}{s}$$

Transfer function from ℓ to y

$$G_{y\ell} = \frac{s}{(s+1)(s-1)}$$

An Remedy

The closed loop is completely characterized by four transfer functions. Look at all four!

$$G(s) = \begin{pmatrix} \dfrac{1}{1+PC} & -\dfrac{C}{1+PC} \\ \dfrac{P}{1+PC} & -\dfrac{PC}{1+PC} \end{pmatrix}$$

If $C(s) = \frac{B_c}{A_c}$ and $P(s) = \frac{B_c}{A_c}$ then

$$G(s) = \begin{pmatrix} \dfrac{A_c A_p}{A_c A_p + B_c B_p} & -\dfrac{A_p B_c}{A_c A_p + B_c B_p} \\ \dfrac{A_c B_p}{A_c A_p + B_c B_p} & -\dfrac{B_p B_c}{A_c A_p + B_c B_p} \end{pmatrix}$$

Alternately that the polynomial $C_{pol} = A_c A_p + B_c B_p$ has no roots in RHP.

Back to the Example

System matrix

$$G(s) = \begin{pmatrix} \dfrac{s}{s+1} & -\dfrac{s-1}{s+1} \\ \dfrac{s}{(s-1)(s+1)} & -\dfrac{1}{s+1} \end{pmatrix}$$

Characteristic polynomial

$$C_{pol} = (s-1)s + s - 1 = (s-1)(s+1)$$

Has a root $s = 1$ in the right half plane.

How to Compare two Systems

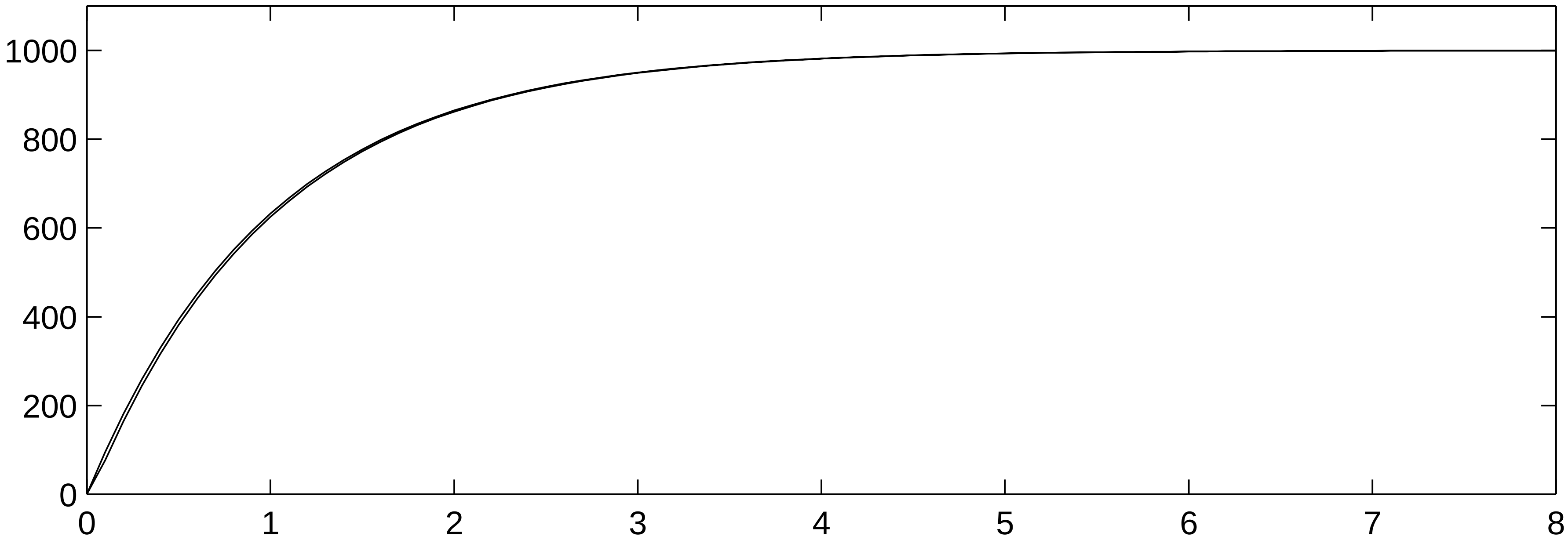
- An important problem
- Related to finding perturbations of a system
- Classically only stable perturbations
- Is this sufficient?
- Mathematically equivalent to finding a norm
- Well known in adaptive control

Similar Open Loop Different Closed Loop

Consider the systems

$$G_1(s) = \frac{1000}{s+1}, \qquad G_2(s) = \frac{1000a^2}{(s+1)(s+a)^2}$$

Step responses



Similar Open Loop Different Closed Loop

Open loop responses

$$G_1(s) = \frac{1000}{s+1}, \qquad G_2(s) = \frac{1000a^2}{(s+1)(s+a)^2}$$

Closed loop transfer functions

$$G_{1cl} = \frac{1000}{s+1001}, \qquad G_{2cl} = \frac{10^7}{(s-287)(s^2+86s+34879)}$$

Notice G_{1cl} stable and G_{2cl} unstable

Different Open Loop Similar Closed Loop

Consider the systems

$$G_1(s) = \frac{1000}{s+1}, \qquad G_2(s) = \frac{1000}{s-1}$$

Notice G_1 stable and G_2 unstable.

With unit gain feedback the closed loop transfer functions become

$$G_{1cl}(s) = \frac{1000}{s+1001} \qquad G_{2cl}(s) = \frac{1000}{s+999}$$

A Long Story

- Stable systems: $\max_\omega |G(i\omega)|$
- Unstable systems:

  Gap metric (Zames)

  Graph metric (Vidyasagar) $G = N/D$

$$\max\left(\max_\omega \left|\frac{D(i\omega)}{N_0(i\omega)}\right|, \max_\omega \left|\frac{D(i\omega)}{N_0(i\omega)}\right|\right)$$

- The Vinnicombe metric (scalar processes)

$$\delta(P_1, P_2) = ||(I + P_2^* P_2)^{-1/2}(P_2 - P_1)(I + P_1^* P_1)^{-1/2}|| < 1$$

if winding number condition satisfied, otherwise $\delta(P_1, P_2) = 1$

The Graph Metric

We know how to compare stable systems. What to do with unstable systems? Let

$$P(s) = \frac{B(s)}{A(s)}$$

where A and B are polynomials. Choose a stable polynomial C whose degree is not lower than the degrees of A and B, then

$$P(s) = \frac{\dfrac{B(s)}{C(s)}}{\dfrac{B(s)}{C(s)}} = \frac{N(s)}{D(s)}$$

Compare the numerator and denominator transfer functions jointly.

Many Ways to Choose C

Two rational functions D and N are called coprime if there exist rational functions X and Y which satisfy the equation

$$XD + YN = 1$$

The condition for coprimeness is essentially that $D(s)$ and $N(s)$ do not have any common factors.

Let $D^*(s) = D(-s)$. A factorization $P = D/N$ such that

$$DD^* + NN^* = 1$$

is called a coprime factorization of P.

Vinnecombe's Metric

Consider two systems with the normalized coprime factorizations

$$P_1 = \frac{D_1}{N_1}, \qquad P_2 = \frac{D_2}{N_2}$$

To compare the systems it must be required that

$$\frac{1}{2\pi} \Delta \arg_\Gamma (N_1 N_2^* + D_1 D_2^*) = 0$$

where Γ is the Nyquist contour. In the polynomial representation this condition implies

$$\frac{1}{2\pi} \Delta \arg_\Gamma (B_1 B_2^* + A_1 A_2^*) = \deg A_2$$

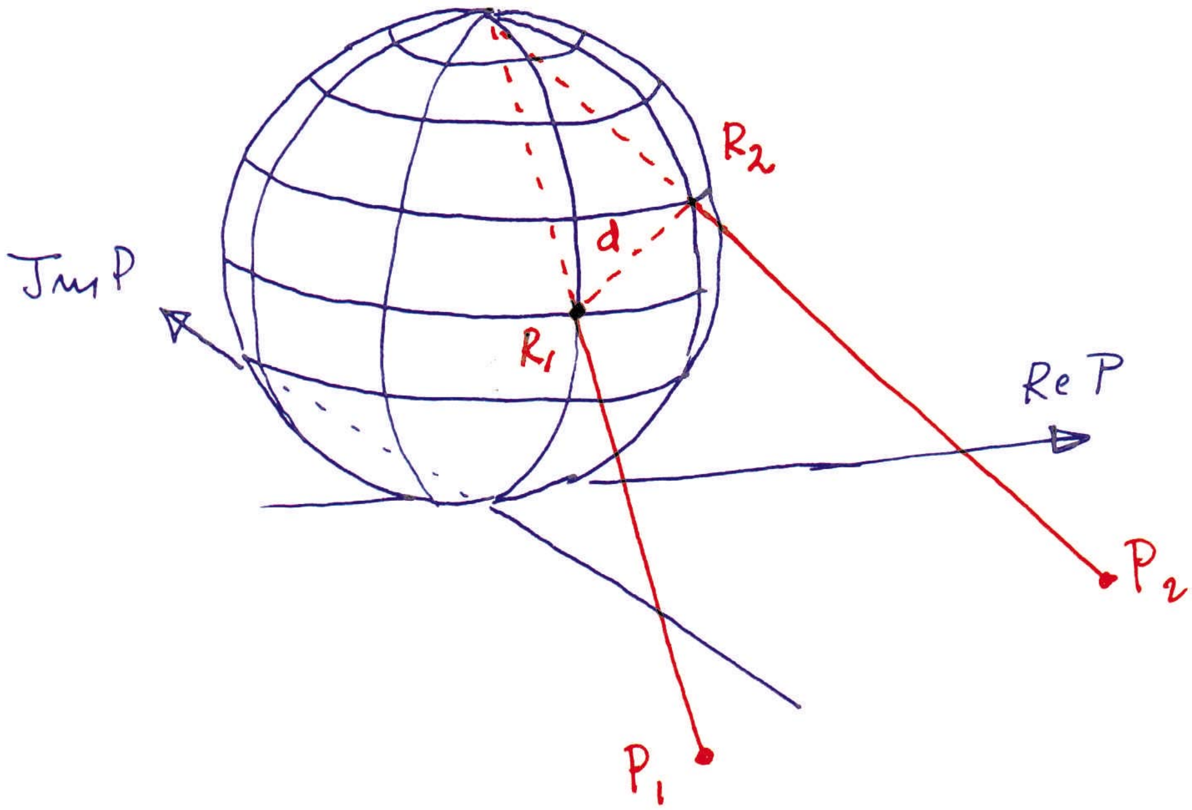
The winding number constraint!

Vinnecombe's Metric

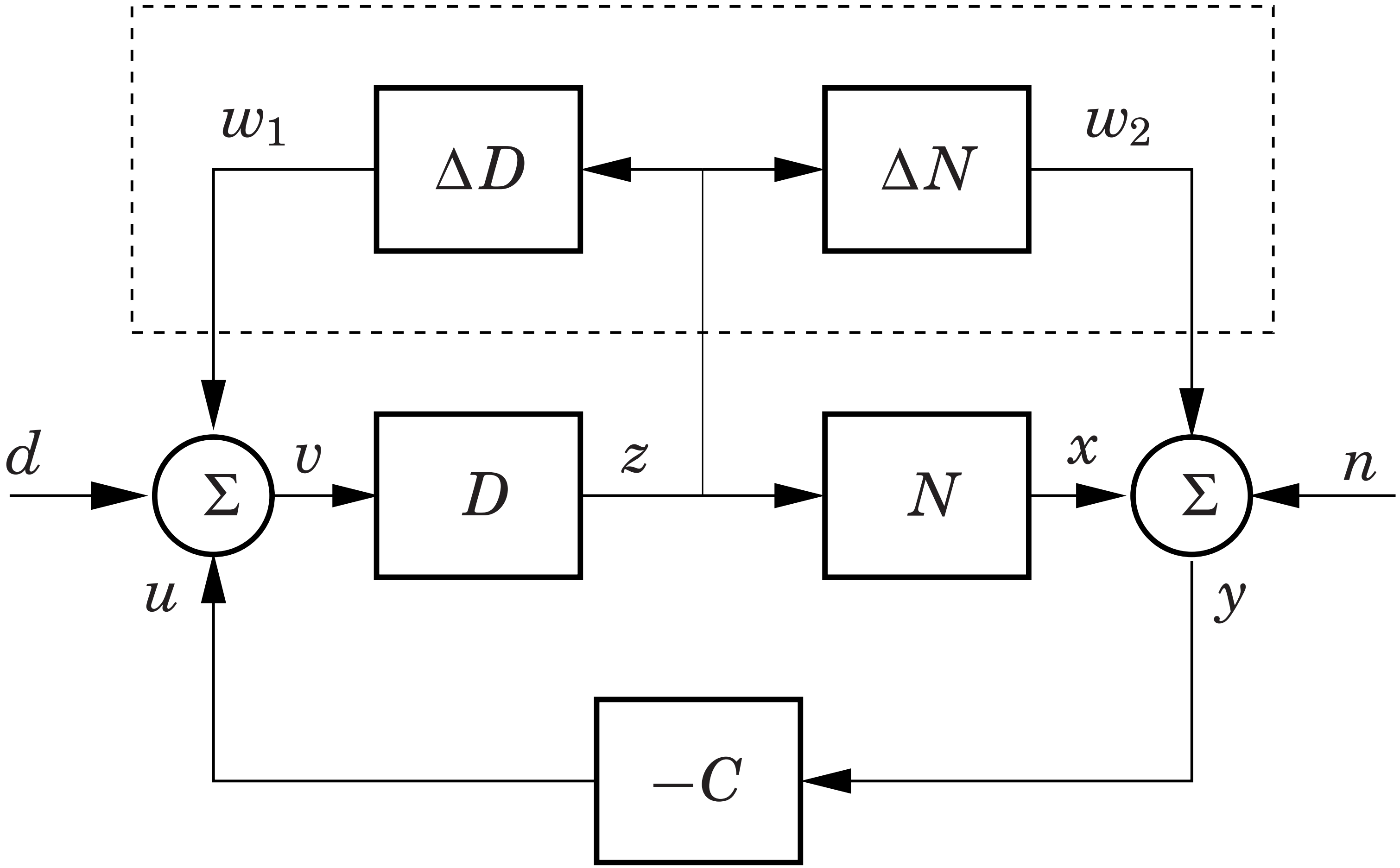
If the winding number constraint is satisfied Vinnecombe's Metric can be defined as

$$\delta_\nu(P_1, P_2) = \sup_\omega \frac{|P_1(i\omega) - P_2(i\omega)|}{\sqrt{(1 + |P_1(i\omega)|^2)(1 + |P_2(i\omega)|^2)}}$$

Geometric Interpretation



Coprime Factor Uncertainty



$$P + \Delta P = \frac{N + \Delta N}{D + \Delta D} = ND^{-1} = D^{-1}N$$

How Large Variations are Permitted?

Straight forward calculations give

$$z = \frac{D^{-1}}{1+PC} w_1 - \frac{D^{-1}C}{1+PC} w_2 = D^{-1} \begin{pmatrix} \frac{1}{1+PC} & -\frac{C}{1+PC} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta D \\ \Delta N \end{pmatrix} z$$

Hence

$$\left\| \begin{pmatrix} \Delta N \\ \Delta D \end{pmatrix} \right\|_\infty \left\| D^{-1} \begin{pmatrix} \frac{1}{1+PC} & -\frac{C}{1+PC} \end{pmatrix} \right\|_\infty < 1$$

Since N, D is a coprime factorization we have

$$\left\| D^{-1} \begin{pmatrix} \frac{1}{1+PC} & -\frac{C}{1+PC} \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} D \\ N \end{pmatrix} D^{-1} \begin{pmatrix} \frac{1}{1+PC} & -\frac{C}{1+PC} \end{pmatrix} \right\|_\infty$$

$$\left\| \begin{pmatrix} I \\ P \end{pmatrix} \begin{pmatrix} \frac{1}{1+PC} & -\frac{C}{1+PC} \end{pmatrix} \right\|_\infty = \|G(P,C)\|_\infty$$

where

$$G(P,C) = \begin{pmatrix} \dfrac{1}{1+PC} & -\dfrac{C}{1+PC} \\ \dfrac{C}{1+PC} & -\dfrac{PC}{1+PC} \end{pmatrix}$$

Introduce the gain of the closed loop system defined as

$$\gamma(P,C) = \sup_\omega \|G(P(i\omega), C(i\omega))\|_\infty$$

Summary

The robustness conditions can be written as follows

$$\left\| \begin{pmatrix} \Delta N \\ \Delta D \end{pmatrix} \right\|_\infty < \frac{1}{\gamma}$$

A nice generalization of the classical result

$$\left| \frac{\Delta P}{P} \right| < \frac{1}{M_t}$$

For scalar systems we have

$$|G(P,C)| = \bar{\sigma} G(P,C) = \frac{\sqrt{(1+|C|^2)(1+|P|^2)}}{|1+PC|}$$

and

$$\gamma(P,C) = \sup_\omega \frac{\sqrt{(1+|C(i\omega)|^2)(1+|P(i\omega)|^2)}}{|1+P(i\omega)C(i\omega)|}$$

Physical Interpretations

The quantity

$$\gamma(P, C) = \sup_{\omega} \|G(P(i\omega), C(i\omega))\|_{\infty}$$

$$G(P, C) = \begin{pmatrix} \dfrac{1}{1+PC} & -\dfrac{C}{1+PC} \\ \dfrac{C}{1+PC} & -\dfrac{PC}{1+PC} \end{pmatrix}$$

is a natural generalization of M_t and M_s. We can therefore call it *the generalized sensitivity*. Vinnecombe defines

$$b(P, C) = \begin{cases} \dfrac{1}{\gamma} & \text{if } (P, C) \text{ is stable,} \\ 0 & \text{otherwise} \end{cases}$$

as the *generalized stability margin*.

Comparison with Ordinary Stability Margin

The generalized stability margin

$$b(P,C) = \begin{cases} \dfrac{1}{\gamma} & \text{if } (P,C) \text{ is stable,} \\ 0 & \text{otherwise} \end{cases}$$

has the property $0 \le b \le 1$. To compare the conventional stability margin could be redefined as

$$A_m^* = 1 - \frac{1}{A_m}$$

Notice that $A_m = 0$ for an unstable system and $A_m = 1$ for a very robust system.

$\mathcal{H}_\infty$ Loop Shaping

This design procedure gives a controller that maximizes the generalized stability margin or equivalently that minimizes the generalized sensitivity.

$$b_{opt} = \sup_C b(P, C)$$

If extremum exists

$$C^* = \arg\max b(P, C)$$

Andrey will teach you how to do this!

Similarities

Vinnecombe's metric

$$\delta_\nu(P_1, P_2) = \sup_\omega \frac{|P_1(i\omega) - P_2(i\omega)|}{\sqrt{(1 + |P_1(i\omega)|^2)(1 + |P_2(i\omega)|^2)}}$$

Norm of the G-matrix

$$\gamma(P, C) = \sup_\omega \frac{\sqrt{(1 + |C(i\omega)|^2)(1 + |P(i\omega)|^2)}}{|1 + P(i\omega)C(i\omega)|}$$

$$\begin{aligned}\frac{1}{\gamma(P, C)} &= \inf_\omega \frac{|1 + P(i\omega)C(i\omega)|}{\sqrt{(1 + |C(i\omega)|^2)(1 + |P(i\omega)|^2)}} \\ &= \inf_\omega \frac{|P(i\omega) + 1/C(i\omega)|}{\sqrt{(1 + (1/|C(i\omega)|)^2)(1 + |P(i\omega)|^2)}}\end{aligned}$$

Hence

Vinnicombe's Theorems

Proposition 1

Consider a nominal processes P, a controller C and a parameter β. The controller C stabilizes all plants P_1 such that $\delta_v(P, P_1) \leq \beta$, if and only if $b(P, C) > \beta$.

Proposition 2

Given a nominal process P, a perturbed process P_1 and a number $\beta < b_{opt}(P, C)$. Then (P_1, C) is stable for all compensators C, such that $b(P_1, C) > \beta$ if and only if $\delta(P, P_1) \leq \beta$.

Connections to Classical Theory

Vinnecombe's theory depends on "the Gang of Four" but classical robustness theory only depends on the loop transfer function PC. Is there a connection?

$$\gamma(P, C) = \sup_{\omega} \frac{\sqrt{(1 + |C(i\omega)|^2)(1 + |P(i\omega)|^2)}}{|1 + P(i\omega)C(i\omega)|}$$

Introduce the weighted process $P' = PW$ and the weighted controller CW^{-1}. The generalized sensitivity γ then becomes

$$\gamma(P', C') = \sup_{\omega} \frac{\sqrt{(1 + |C(i\omega)W^{-1}(i\omega)|^2)(1 + |P(i\omega)W(i\omega)|^2)}}{|1 + P(i\omega)C(i\omega)|}$$

Is there a best weighting? Yes $W = \sqrt{|C|/|P|}$ gives

$$\gamma^* = \sup_{\omega}(|\frac{1}{1 + PC(i\omega)}| + |\frac{PC(i\omega)}{1 + PC(i\omega)}|)$$

Contour of Constant Gamma

The contour of constant γ can be represented as follows

$$r(\varphi) = -\frac{\gamma^2 \cos\varphi - 1}{\gamma^2 - 1} \pm \sqrt{\left(\frac{\gamma^2 \cos\varphi - 1}{\gamma^2 - 1}\right)^2 - 1}.$$

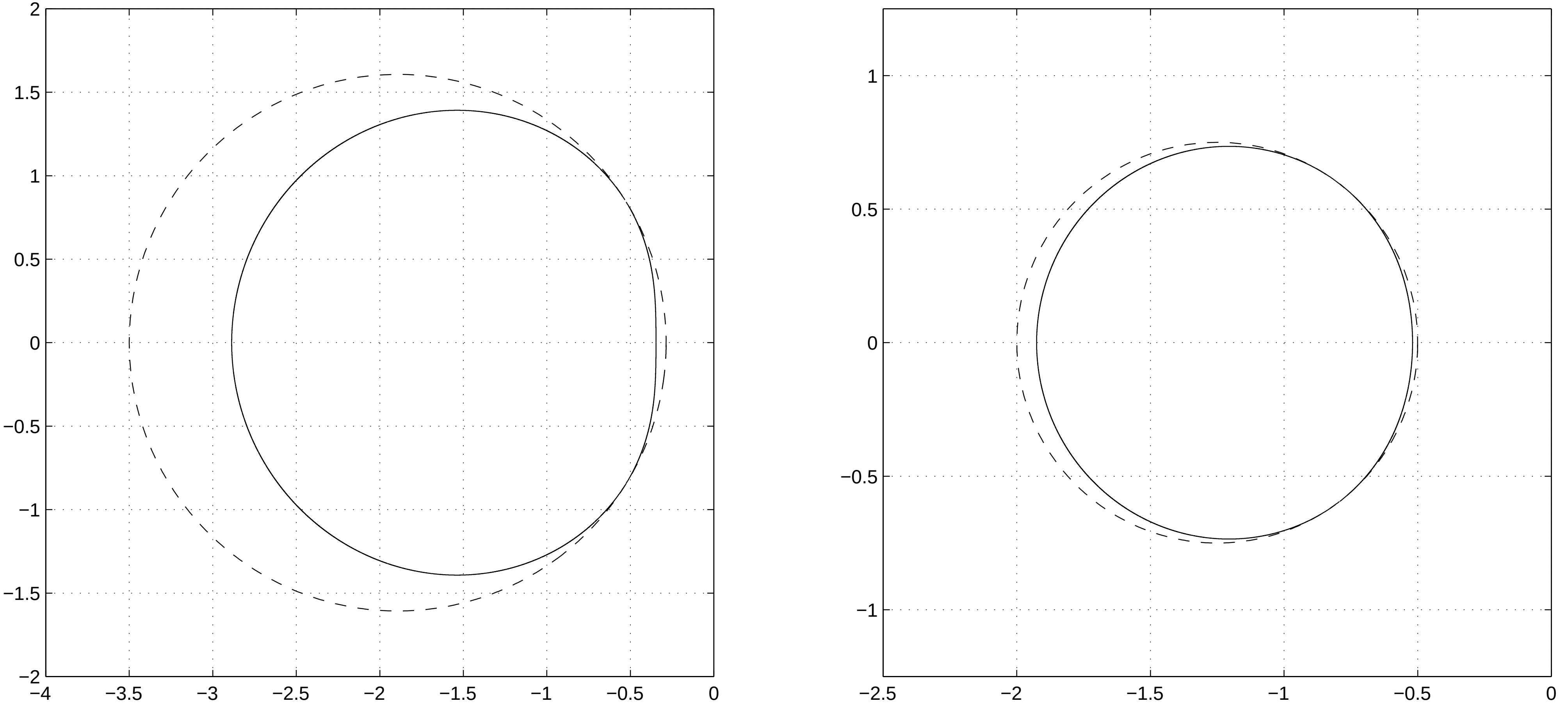
Can be enclosed in combined M_s, M_p circles, with center C and radius R

$$C = -\frac{2M^2 - 2M + 1}{2M(M-1)}, \qquad R = \frac{2M - 1}{2M(M-1)}.$$

M_+	1.37	1.40	1.50	1.60	1.65	1.70	1.80	1.90	1.91	2.00
M_-	1.50	1.53	1.62	1.71	1.75	1.80	1.90	1.99	2.00	2.08
γ	2.00	2.06	2.24	2.42	2.50	2.60	2.79	2.97	3.00	3.16

Contour of Constant Gamma

$$M_+ = \frac{1}{2}\left(1 + \sqrt{\gamma^2 - 1}\right), \qquad M_- = \frac{1}{2}\left(1 + \gamma\right).$$



Conclusions

- Early frequency domain methods took process uncertainty explicitly into account. The methods were difficult to extend to systems with many inputs and many outputs. Not computationally tractable.
- State space theory gave new views on the problem and gave good design techniques but robustness was not explicitly dealt with. Necessary to check robustness and to be aware of fundamental limitations.
- $\mathcal{H}_\infty$ theory brought robustness back into the main stream
- $\mathcal{H}_\infty$ loop-shaping puts many ideas together
- so far no good generalization of delay margin