## Robustness analysis and quadratic inequalities

- $\mu$-analysis
- Multipliers
- S-procedure
- Integral Quadratic Constraints
- Performance analysis

## Material

- lecture notes
- A. Megretski and A. Rantzer, System Analysis via Integral Quadratic Constraints, IEEE Transactions on Automatic Control, 47:6, 1997
- U. Jönsson, Lecture Notes on Integral Quadratic Constraints
- User's guide to $\mu$—toolbox, Matlab

## Preview — Example

A linear system of equations

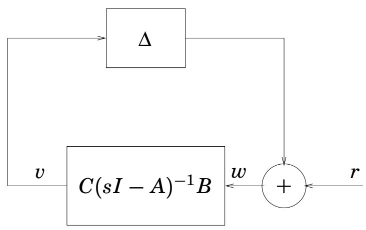
$$\begin{cases} x = y \\ y = 1.1 - 0.1x \end{cases} \qquad \Rightarrow \qquad x = y = 1$$

Equations with uncertainty

$$\begin{cases} (x-y)^2 < \epsilon_1 x^2 \\ (y + 0.1x - 1.1)^2 < \epsilon_2 \end{cases} \qquad \Rightarrow \quad (x-1)^2 + (y-1)^2 < \epsilon_3$$

Given $\epsilon_1$ and $\epsilon_2$, how do we find a valid $\epsilon_3$?

## Example

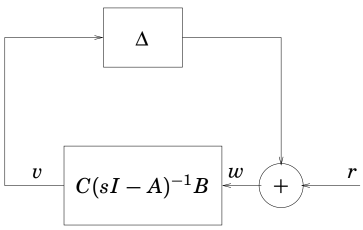



Question: For what values of $\Delta$ is the system stable?
Note: May be large differences if we consider complex or real uncertainties $\Delta$.

## Parametric Uncertainty in Linear Systems

Let $\mathcal{D} \subset \mathbf{R}^{n\times n}$ contain zero. The system $\dot{x} = (A + B\Delta C)x$ is then exponentially stable for all $\Delta \in \mathcal{D}$ if and only if

- A is stable
- $\det\left[I - \Delta C(i\omega I - A)^{-1}B\right] \neq 0$ for $\omega \in \mathbf{R}$, $\Delta \in \mathcal{D}$




## Use quadratic inequalities at each frequency!

$$w = \left[I - \Delta C(i\omega I - A)^{-1}B\right]^{-1} r$$

$$\begin{cases} w = \Delta v + r \\ v = C(i\omega I - A)^{-1}Bw \end{cases}$$

For example, if

$$\mathcal{D} = \left\{ \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} : \quad \delta_k \in [-1, 1] \right\}$$

Then a bound of the form $|w|^2 < \gamma^2 |r|^2$ can be obtained using

$$\begin{matrix} |w_1 - r_1|^2 < |v_1|^2 \\ |w_2 - r_2|^2 < |v_2|^2 \end{matrix} \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C(i\omega I - A)^{-1}B \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

This verifies that $\det\left[I - \Delta C(i\omega I - A)^{-1}B\right] \neq 0$.

## Structured Singular Values

Given $M \in \mathbf{C}^{n\times n}$ and a perturbation set

$$\mathcal{D} = \{\mathrm{diag}[\delta_1 I_{r_1}, \ldots, \delta_m I_{r_m}, \Delta_1, \ldots, \Delta_p] : \delta_k \in \mathbf{R}, \Delta_l \in \mathbf{C}^{m_l \times m_l}\}$$

the **structured singular value** $\mu_{\mathcal{D}}(M)$ is defined by

$$\mu_{\mathcal{D}}(M) = \sup\{\bar{\sigma}(\Delta)^{-1} : \Delta \in \mathcal{D}, \det(I - M\Delta) = 0\}$$

See Matlab's $\mu - toolbox$

## Reformulated Definition

The following two conditions are equivalent

(*i*) $0 \neq \det[I - \Delta M(i\omega)]$ for all $\Delta \in \mathcal{D}$ and $\omega \in \mathbf{R}$
(*ii*) $\mu_{\mathcal{D}}(M(i\omega)) < 1$ for $\omega \in \mathbf{R}$

## Bounds on $\mu$

If $\mathcal{D}$ consists of full complex matrices, then $\mu_{\mathcal{D}}(M) = \bar{\sigma}(M)$.

where $\bar{\sigma}(M)$ is the largest singular value of M = the larges eigenvalue of the matrix $M^*M$.

If $\mathcal{D}$ consists of perturbations of the form $\Delta = \delta I$ with $\delta \in [-1, 1]$, then $\mu_{\mathcal{D}}(M)$ is equal to the magnitude $\rho_{\mathbf{R}}(M)$ of the largest real eigenvalue of $M$ ("the spectral radius"). In general

$$\rho_{\mathbf{R}}(M) \leq \mu_{\mathcal{D}}(M) \leq \bar{\sigma}(M)$$

## S-procedure

Let $M_0$, $M_1$, $\ldots M_p$ be quadratic functions of $z \in R^n$

$$M_i = z^T T_i z + 2u_i z + v_i, \quad i = 0, \ldots, p$$

where $T_i = T_i^T$.

Consider the following condition on $M_0$, $M_1$, $\ldots M_p$:

$$M_0(z) \leq 0 \quad \text{for all } z \ \underline{\text{such that}} \quad M_i(z) \geq 0, \quad i = 1, \ldots, p \qquad (1)$$

## Computation of $\mu$

Define

$$\mathcal{U}_{\mathcal{D}} = \{U \in \mathcal{D} : U'U = I\}$$
$$\mathcal{D}_{\mathcal{D}} = \{D = D' \in \mathbf{C}^n : D\Delta = \Delta D \text{ for all } \Delta \in \mathcal{D}\}$$
$$\mathcal{G}_{\mathcal{D}} = \{G = G' \in \mathbf{C}^n : G\Delta = \Delta' G \text{ for all } \Delta \in \mathcal{D}\}$$

Then

$$\sup_{U \in \mathcal{U}_{\mathcal{D}}} \rho_{\mathbf{R}}(UM) \leq \mu_{\mathcal{D}}(M) \leq \inf_{\substack{D \in \mathcal{D}_{\mathcal{D}} \\ G \in \mathcal{G}_{\mathcal{D}}}} \hat{\mu}(D, G) \leq \inf_{D \in \mathcal{D}_{\mathcal{D}}} \bar{\sigma}(DMD^{-1})$$

where

$$\hat{\mu}(D, G) = \inf\{\mu > 0 : M'D'DM + j(GM - M'G) < \mu^2 D'D\}$$

Consider the following condition on $M_0$, $M_1$, $\ldots M_p$:

$M_0(z_*) \leq 0$ $\quad$ <u>for all $z_*$ such that</u> $\quad M_i(z_*) \geq 0, \quad i = 1, \ldots, p$
(1)

Obviously,

if there exists $\tau_1 \geq 0, \ldots \tau_p \geq 0$ such that for all $z$

$$M_0(z) + \sum_{i=1}^{p} \tau_i M_i(z) \leq 0 \qquad (2)$$

then (1) holds.

Nontrivial fact, that when $p = 1$ , (1) implies (2), provided that there exist some $z_0$ such that $M_0(z_0) < 0$.

## S-procedure for quadratic inequalities

The inequality

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T M_0 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \leq 0$$

follows from the inequalities

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T M_1 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq 0 \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T M_2 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq 0$$

if there exist $\tau_1, \tau_2 \geq 0$ such that

$$M_0 + \tau_1 M_1 + \tau_2 M_2 \leq 0$$

Numerical algorithms in available (e.g. in Matlab), see also [Boyd *et al*]

## S-procedure losslessness by Megretski/Treil

Let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be time-invariant quadratic forms on $\mathbf{L}_2^m$. Suppose that there exists $z_*$ such that

$$\sigma_1(z_*) > 0, \ldots, \sigma_m(z_*) > 0$$

Then the following statements are equivalent

- $\sigma_0(z) \leq 0$ for all $z$ such that $\sigma_1(z) \geq 0, \ldots, \sigma_n(z) \geq 0$
- There exist $\tau_1, \ldots, \tau_n \geq 0$ such that

$$\sigma_0(z) + \sum_k \tau_k \sigma_k(z) \leq 0 \qquad \forall z$$

## S-procedure in general

The inequality

$$\sigma_0(h) \leq 0$$

follows from the inequalities

$$\sigma_1(h) \geq 0, \ldots, \sigma_n(h) \geq 0$$

if there exist $\tau_1, \ldots, \tau_n \geq 0$ such that

$$\sigma_0(h) + \sum_k \tau_k \sigma_k(h) \leq 0 \qquad \forall h$$

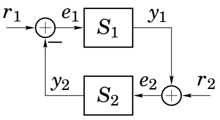
## Integral Quadratic Constraint

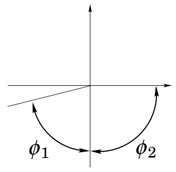
An IQC express information of a subsystem. Should be convenient to use for analysis of a larger system.

Unifies

- Multiplier (Zames-Falb)
- Passivity
- Absolute stability
- $\mu$

## Passivity Theorem is a “Small Phase Theorem”



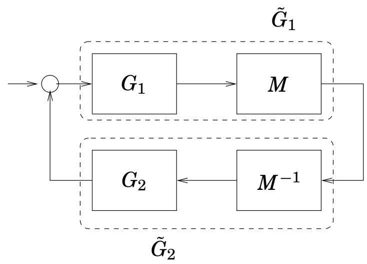




A passive operator can also be viewed as a sector condition $[0, \infty)$.

Compare circle criterion: Nyquist curve should avoid "circle" $(-\frac{1}{\alpha}, -\frac{1}{\beta}) \to whole, LHP$ as $\alpha \to 0$ and $\beta \to \infty$.

Cut the loops in smart ways ($H_1$ and $H_2$) or introduce multipliers (Bounded operators $M$ and $M^{-1}$).

Same idea as loop transformations: should be easier to prove stability for transformed system.




## Multipliers, cont’d

Example: Negative feedback of linear system

$$G(s) = C(sI - A)^{-1}B$$

with nonlinearity with positivity property $\varphi(y) \cdot y \geq 0$.

Circle criterion assures exponential stability if

$$\text{Re}\{G(i\omega)\} > 0 \quad \omega \in R$$

Compare (strict) passivity conditions

## Multipliers, cont’d

Zames-Falb (1968):

Circle criterion can be improved if there are additional assumptions on the nonlinearity as *e.g.,* monotonicity or bounds on slope.

If $\dfrac{d\varphi(y)}{dy} \geq 0$, Z-F introduced extra freedom with

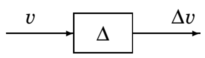
$$H \in RL_\infty \text{ and } ||H||_{L_1} \leq 1$$

such that absolute stability is assured if

$$\text{Re}\{G(i\omega)^{-1} \cdot (1 + H(i\omega))\} > 0, \quad \omega \in R\ \{0\}$$

Compare with Popov-criterion conditions

## Integral Quadratic Constraint




The causal bounded operator $\Delta$ on $\mathbf{L}_2^m$ is said to satisfy the IQC defined by the matrix function $\Pi(i\omega)$ if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega \quad \geq \quad 0$$

for all $v \in \mathbf{L}_2$.

Trivial for $\Pi > 0$, but almost all interesting cases have non-positive $\Pi$.

## Example — Gain and Passivity

Suppose the gain of $\Delta$ is at most one. Then

$$0 \leq \int_0^\infty (|v|^2 - |\Delta v|^2)dt = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega$$

Suppose instead that $\Delta$ is passive. Then

$$0 \leq \int_0^\infty v(t)(\Delta v)(t)dt = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega$$

Note: Scaling in Parseval’s formula neglected here (does not affect sign of IQC).

## Exercise

Show that a nonlinearity satisfying the sector condition

$$\alpha y^2 \leq \varphi(t, y)y \leq \beta y^2$$

satisfies the IQC, $\varphi \in IQC(\Pi)$ given by

$$\Pi(j\omega) = \Pi = \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}$$

Note: Satisfies a quadratic inequality (for every frequency) $\Longrightarrow$ satisfies integral quadratic inequality

## IQC’s for Coulomb Friction

$$\begin{cases} f(t) = -1 & \text{if } v(t) < 0 \\ f(t) \in [-1, 1] & \text{if } v(t) = 0 \\ f(t) = 1 & \text{if } v(t) > 0 \end{cases}$$

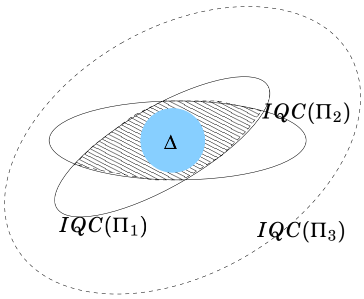
Zames/Falb’s property

$$0 \leq \int_0^\infty v(t)[f(t) + (h * f)(t)]dt, \qquad \int_{-\infty}^{\infty} |h(t)|dt \leq 1$$

$$0 \leq \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v} \\ \widehat{f} \end{bmatrix}^* \begin{bmatrix} 0 & 1 + H(i\omega) \\ 1 + H(-i\omega) & 0 \end{bmatrix} \begin{bmatrix} \widehat{v} \\ \widehat{f} \end{bmatrix} d\omega$$

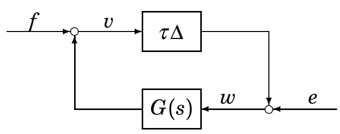
| $\Delta$ structure | $\Pi(i\omega)$ | Condition |
|---|---|---|
| $\Delta$ passive | $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ | |
| $\lVert\Delta(i\omega)\rVert \le 1$ | $\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$ | $x(i\omega) \ge 0$ |
| $\delta \in [-1,1]$ | $\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$ | $X = X^* \ge 0$, $Y = -Y^*$ |
| $\delta(t) \in [-1,1]$ | $\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$ | |
| $(\Delta v)(t) = \mathrm{sgn}(v(t))$ | $\begin{bmatrix} 0 & 1+H(i\omega) \\ 1+H(i\omega)^* & 0 \end{bmatrix}$ | $\lVert H\rVert_{\mathbf{L}_1} \le 1$ |

Use as many IQCs as possible to characterize the nonlinearity/uncertainty.




In this case $IQC(\Pi_3)$ does not help to restrict the complete set that satisfy the IQCs.

## Well-posed Interconnection

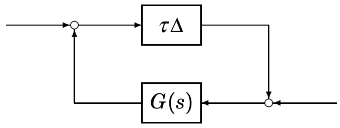



The feedback interconnection

$$\begin{cases} v &= Gw + f \\ w &= \Delta(v) + e \end{cases}$$

is said to well-posed if the map $(v,w) \mapsto (e,f)$ has a causal inverse. It is called BIBO stable if the inverse is also bounded.

## IQC Stability Theorem




Let $G(s)$ be stable and proper and let $\Delta$ be causal.

For all $\tau \in [0,1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

then the feedback system is BIBO stable.

## Computations via LMI's

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \sum_k \tau_k \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

$$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* (M + \sum_k \tau_k M_k) \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

$$\sum_{i=1}^n \tau_k M_k + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} < 0.$$

Solve for $\tau_1, \ldots, \tau_n \ge 0$ and $P$.

## Special Case — $\mu$ Analysis

Note that $\Delta = \mathrm{diag}\{\delta_1, \ldots, \delta_m\}$, with $|\delta_k| \le 1$ satisfies the IQC defined by

$$\Pi(i\omega) = \begin{bmatrix} X(i\omega) & 0 \\ 0 & -X(i\omega) \end{bmatrix}$$

where $X(i\omega) = \mathrm{diag}\{x_1(i\omega), \ldots, x_m(i\omega)\} > 0$.

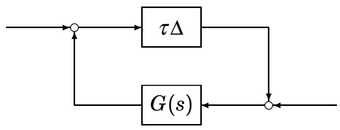
Feedback loop stability follows if there exists $X(i\omega) > 0$ with

$$G(i\omega)^* X(i\omega) G(i\omega) < X(i\omega) \qquad \omega \in [0,\infty]$$

or equivalently, with $D(i\omega)^* D(i\omega) = X(i\omega)$

$$\sup_\omega \lVert D(i\omega) G(i\omega) D(i\omega)^{-1} \rVert < 1$$

## Relation to Passivity and Gain Theorems




A stability theorem based on gain is recovered with $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

A passivity based stability theorem is recovered with $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

## Combination of Uncertain and Nonlinear Blocks

The operator $\Delta(v_1, v_2) = (\delta v_1, \phi(v_2))$ where

$$\delta \in [-1,1]$$
$$\alpha \le \phi(v_2)/v_2 \le \beta$$

satisfies all IQC's defined by matrix functions of the form

$$\Pi(i\omega) = \begin{bmatrix} X(i\omega) & 0 & Y(i\omega) & 0 \\ 0 & -2\alpha\beta & 0 & \alpha+\beta \\ Y(i\omega)^* & 0 & -X(i\omega) & 0 \\ 0 & \alpha+\beta & 0 & -2 \end{bmatrix}$$

where $X(i\omega) = X(i\omega)^*$ and $Y(i\omega) = Y(i\omega)^*$.

## Proof idea of IQC Theorem

Combination of the IQC for $\Delta$ with the inequality for $G$ gives existence of $c_0 > 0$ such that

$$\|v\| \le c_0 \|v - \tau G\Delta(v)\| \qquad v \in \mathbf{L}_2, \tau \in [0,1]$$

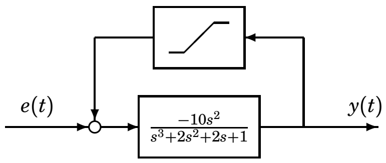
If $(I - \tau G \Delta)^{-1}$ is bounded for some $\tau \in [0,1]$ then the above inequality gives boundedness of $(I - \nu G\Delta)^{-1}$ for all $\nu$ with

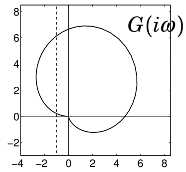
$$c_0 \|G\Delta\| \cdot |\tau - \nu| < 1$$

Hence, boundedness for $\tau = 0$ gives boundedness for $\tau < (c_0\|G\Delta\|)^{-1}$. This, in turn, gives boundedness for $\tau < 2(c_0\|G\Delta\|)^{-1}$ and so on. Finally the whole interval $[0,1]$ is covered.

## A toolbox for IQC analysis

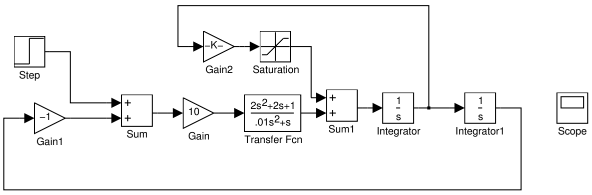
Copy `/home/kursolin/matlab/lmiinit.m` to the current directory or download and install the IQCbeta toolbox from `http://www.ee.mu.oz.au/staff/cykao/`



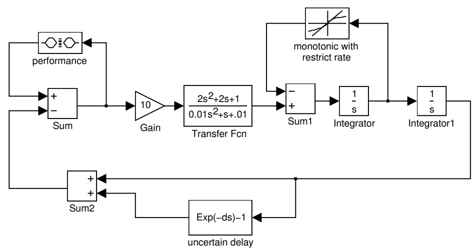




```
>> abst_init_iqc;
>> G = tf([10 0 0],[1 2 2 1]);
>> e = signal
>> w = signal
>> y = -G*(e+w)
>> w==iqc_monotonic(y)
>> iqc_gain_tbx(e,y)
```

## A simulation model




## An analysis model defined graphically

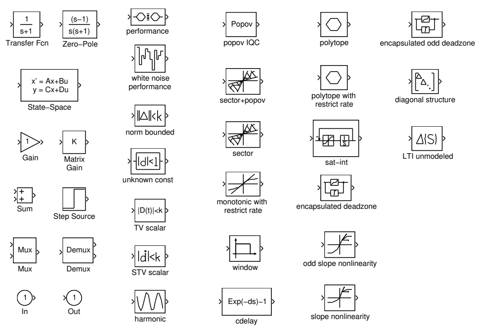



The text version (i.e., NOT the gui) is strongly recommended by the IQCbeta author(s) at present version!!

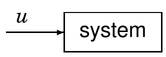
```
ż iqc_gui('fricSYSTEM')

extracting information from fricSYSTEM ...

    scalar inputs:  5
    states:         10
    simple q-forms: 7

    LMI #1    size = 1    states: 0
    LMI #2    size = 1    states: 0
    LMI #3    size = 1    states: 0
    LMI #4    size = 1    states: 0
    LMI #5    size = 1    states: 0

  Solving with 62 decision variables ...

  ans =     4.7139
```

The text version (i.e., NOT the gui) is strongly recommended by the IQCbeta author(s) at present version!!

## A library of analysis objects




## Bounds on Auto Correlation



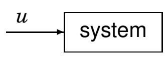

The auto correlation bound

$$\int_{-\infty}^{\infty} u(t)^* u(t-T)dt \quad \le \quad \alpha \int_{-\infty}^{\infty} u(t)^* u(t)dt,$$

corresponds to

$$\Psi(i\omega) = 2\alpha - e^{i\omega T} - e^{-i\omega T}.$$

## Dominant Harmonics

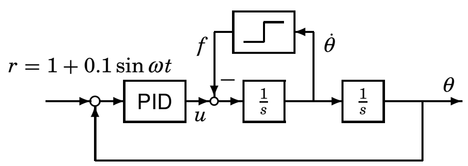


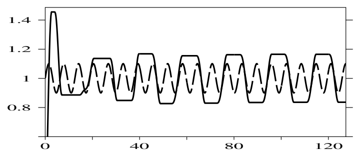

For small $\epsilon > 0$, the constraint

$$\int_0^{\infty} |\hat{u}(i\omega)|^2 d\omega \quad \le \quad (1+\epsilon) \int_a^b |\hat{u}(i\omega)|^2 d\omega$$

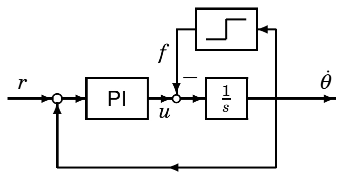
means that the energy of $u$ is concentrated to the interval $[a, b]$.

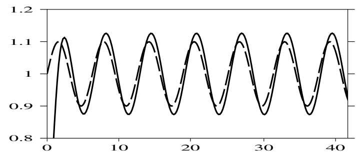
## Subharmonic Oscillations in Position Control



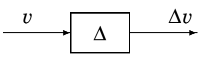



## No Subharmonics in Velocity Control!



## Incremental Gain and Passivity



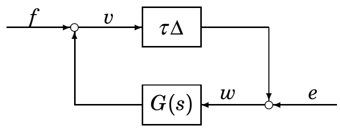

A causal nonlinear operator $\Delta$ on $\mathbf{L}_2^m$ is said to have **incremental gain** less than $\gamma$ if

$$\|\Delta(v_1) - \Delta(v_2)\| \le \gamma \|v_1 - v_2\| \qquad v_1, v_2 \in \mathbf{L}_2$$

It is called **incrementally passive** if

$$0 \le \int_0^T [\Delta(v_1) - \Delta(v_2)][v_1 - v_2]dt \qquad T > 0, v_1, v_2 \in \mathbf{L}_2$$

## Incremental Stability



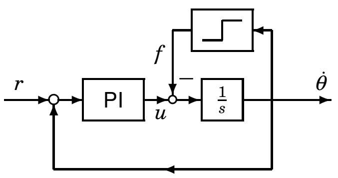

The feedback interconnection

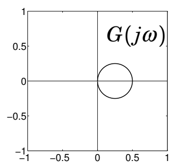
$$\begin{cases} v &= Gw + f \\ w &= \Delta(v) + e \end{cases}$$

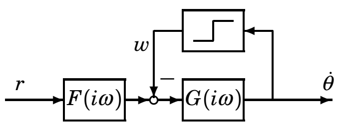
is called **incrementally stable** if there is a constant $C$ such that any two solutions $(e_1, f_1, v_1, w_1)$, $(e_2, f_2, v_2, w_2)$ satisfies

$$\|v_1 - v_2\| + \|w_1 - w_2\| \le C\|e_1 - e_2\| + C\|f_1 - f_2\|$$

## Incremental Stability Excludes Subharmonics

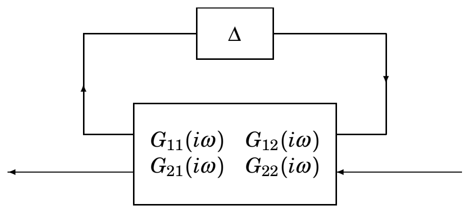



$G(s) = \frac{s}{s^2 + Ks + K_i}$

## Robust Performance




## A Converse Small Gain Theorem

**The static case**
A matrix $M$ satisfies $\bar\sigma(M) < 1$ if and only if $0 \neq \det(I - \Delta M)$ for all matrices $\Delta$ with $\bar\sigma(\Delta) \le 1$.

**The dynamic case**
A stable transfer matrix $G(s)$ satisfies $\|G\|_\infty < 1$ if and only if $[I - \Delta(s)G(s)]^{-1}$ is stable for every stable $\Delta(s)$ with $\|\Delta\|_\infty < 1$.

## Proof in the Static Case

Suppose that $\det(I - \Delta M) = 0$ and $\bar\sigma(\Delta) \le 1$. Then there exists $x \neq 0$ such that $x - \Delta M x = 0$ and

$$|x| = |\Delta M x| \le |Mx|$$

so $\bar\sigma(M) \ge 1$.

On the other hand, if $\bar\sigma(M) \ge 1$, there exists $x \neq 0$ with $|Mx| \ge |x|$. Let

$$\Delta = \frac{xM'x'}{|Mx|^2}$$

Then $\bar\sigma(\Delta) \le 1$ and $(I - \Delta M)x = 0$, so $\det(I - \Delta M) = 0$.

## Robust Performance Theorem

Suppose that $\bar\sigma(M_{11}) \le 1$ and that $\mathcal{D}$ is a connected set of matrices with $0 \in \mathcal{D}$. Let $\mathcal{D}_1 = \{\text{diag}(\Delta_1, \Delta) : \bar\sigma(\Delta_1) \le 1, \Delta \in \mathcal{D}\}$. Then the following conditions are equivalent.

$(i)$ $\bar\sigma(M_{11} + M_{12}[I - \Delta M_{22}]^{-1}\Delta M_{21}) < 1$ for $\Delta \in \mathcal{D}$

$(ii)$ $0 \neq \det\left(I - \begin{bmatrix}\Delta_1 & 0\\ 0 & \Delta\end{bmatrix}\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\right)$ for $\begin{cases}\Delta \in \mathcal{D}\\ \bar\sigma(\Delta_1) \le 1\end{cases}$

$(iii)$ $\mu_{\mathcal{D}_1}\left(\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\right) \le 1$ for $\omega \in \mathbf{R}$

## Proof

The conditions $(ii)$ and $(iii)$ are equivalent by the definition of $\mu$. Moreover, we showed on the previous slide that $(i)$ fails if and only if there exists $\Delta_1$ with $\bar\sigma(\Delta_1) \le 1$ and $x_1 \neq 0$ such that

$$0 = \{I - \Delta_1(M_{11} + M_{12}[I - \Delta M_{22}]^{-1}\Delta M_{21})\}x_1$$

Introduce $x_2 = [I - \Delta M_{22}]^{-1}\Delta M_{21}x_1$. Then

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\Delta_1 & 0\\ 0 & \Delta\end{bmatrix}\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

This is possible if and only if $(ii)$ fails, so the equivalence of $(i)$ and $(ii)$ is proved.

## Example

Compute

$$\max_{|\delta_k| \le 1} \sup_\omega \frac{\delta_1}{(i\omega)^2 + (2+\delta_2)i\omega + 2 + \delta_1\delta_2}$$

This is the worst case gain of the system

$$\begin{cases}\ddot y = -(2+\delta_2)\dot y - (2+\delta_1\delta_2)y + \delta_1 u = -2\dot y - 2y - \delta_1 v_1 - \delta_2 v_2\\ v_1 = -\delta_2 v_3 + u, \qquad v_2 = \dot y, \qquad v_3 = y\end{cases}$$

$$\begin{bmatrix}\dot y\\ \ddot y\\ v_1\\ v_2\\ v_3\\ y\end{bmatrix} = \left[\begin{array}{cc|cccc}0 & 1 & 0 & 0 & 0 & 0\\ -2 & -2 & 1 & -1 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & -1 & 1\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{array}\right]\begin{bmatrix}y\\ \dot y\\ \delta_1 v_1\\ \delta_2 v_2\\ \delta_2 v_3\\ u\end{bmatrix}$$

## Performance Analysis via S-procedure

The performance criterion

$$\sigma_0(h) \le 0 \qquad \forall h \in \mathcal{N}$$

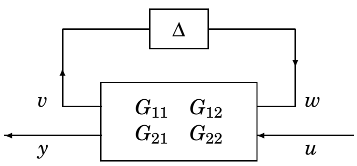
follows from the IQC's

$$\sigma_1(h) \ge 0, \dots, \sigma_n(h) \ge 0 \qquad \forall h \in \mathcal{N}$$

if there exist $\tau_1, \dots, \tau_n \ge 0$ such that

$$\sigma_0(h) + \sum_k \tau_k\sigma_k(h) \le 0 \qquad \forall h \in \mathbf{L}_2^n$$

## Performance Bounds from IQC's




Suppose that $\Delta$ satisfies the IQC defined by $\Pi$. Then the gain bound $\|y\| \le \gamma\|u\|$ holds provided that the system is stable and

$$0 \ge \begin{bmatrix}G(i\omega)\\ I\end{bmatrix}^* \begin{bmatrix}\Pi_{11} & 0 & \Pi_{12} & 0\\ 0 & I & 0 & 0\\ \Pi_{21} & 0 & \Pi_{22} & 0\\ 0 & 0 & 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}G(i\omega)\\ I\end{bmatrix} \qquad \omega \in \mathbf{R}$$

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User's guide to $\mu$—toolbox, Matlab

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