Hand-in problems due Fri October 20

1. Absolute stability, Kalman-Yakubovich-Popov Lemma, The Circle and Popov criteria, S-procedure, μ-analysis, IQCs, performane analysis

Reading assignment

Lecture notes, Khalil (3rd ed.) Chapters 6, 7.1.
Extra material on the K-Y-P Lemma (paper by Rantzer), see the lecture notes for ref to IQCbeta-toolbox (Matlab) and Ulf's lecture notes (handout), Matlab μ-tools manual. Megretski/Rantzer IEEE TAC 47:6, 1997 (or the Technical reports). IQCbeta manual.

1.1 Comments on the text

This section of the book presents some of the core material of the course. The results have played a central role in control theory for a long time and have recently been vitalized by new progress, both in theory and in computational methods.

The concept *absolute stability* is introduced for nonlinear systems consisting of two parts, one linear time-invariant and one nonlinear. Detailed knowledge about the nonlinear part is not used, only inequality constraints.

The Kalman-Yakubovich-Popov Lemma shows that a transfer function inequality is equivalent to a condition on solvability of a linear matrix inequality (LMI) defined by the state space matrices. In the proof of the circle and Popov criteria, the LMI appears naturally in the attempt to construct a Lyapunov function. The K-Y-P Lemma therefore connects the existence of a certain Lyapunov function to a transfer function condition on the linear part. Khalil does not provide a complete proof, instead we refer to separate notes which are distributed this week.

Recently, the same lemma has often been used in the opposite direction, as frequency conditions on multivariable transfer functions are verified by translating them into an LMI condition, which can be solved by convex optimization. Some of the exercises below will illustrate this and the MATLAB Toolbox IQCbeta (here used as wrapper to LMI-lab) will be useful for the calculations.

Soon after the appearance of the Popov criterion, for example in the textbook by Aiserman and Gantmacher from 1965, it was pointed out that the Popov criterion holds also with negative slope $1/\eta$ on the Popov line. However, this fact is ignored by Khalil and several other western textbooks. Can you see why it must be true?

Exercise 3.1 = Kha. 7.3

Exercise 3.2 = Kha. 7.4

Exercise 3.3 = Kha. 7.1 (1),(2),(3)

Exercise 3.4 Solve the previous exercise with the circle criterion replaced by the Popov criterion.

Exercise 3.5 In the first problem set we considered subproblem **(a)** below.

a. Find a quadratic simultaneous Lyapunov function (for example using LMI-lab or yalmip) for the linear time-varying system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & -8 \\ 6 & -13 \end{bmatrix} x + \begin{bmatrix} 8 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} \delta_1(t) & 0 \\ 0 & \delta_2(t) \end{bmatrix} y \\ y &= \begin{bmatrix} 1 & -1 \\ -11 & 16 \end{bmatrix} x \end{aligned}$$

where $|\delta_k(t)| \leq 1$, $k = 1, 2$.

b. Does Theorem 7.1 prove stability if the δ-matrix is replaced by a memoryless nonlinearity satisfying the sector condition

$$[\psi(t,y) + y]^T [\psi(t,y) - y] \leq 0, \quad \forall t \geq 0, y \in \mathbf{R}^2$$

Exercise 3.6 Consider a $p \times p$ matrix function $M(s)$, which is analytic for $\operatorname{Re} s > 0$ and satisfies $M(s) = \overline{M(\bar{s})}$. The matrix function is called

output strictly passive (OSP) if $\exists \epsilon > 0$ such that for $\operatorname{Re} s > 0$

$$M(s) + M(s)^* \geq \epsilon M(s)^* M(s)$$

input strictly passive (ISP) if $\exists \epsilon > 0$ such that for $\operatorname{Re} s > 0$

$$M(s) + M(s)^* \geq \epsilon$$

positive real (PR) if for $\operatorname{Re} s > 0$

$$M(s) + M(s)^* \geq 0$$

strictly positive real (SPR) if $\exists \epsilon > 0$ such that $M(s - \epsilon)$ is PR.

a. For the scalar transfer function $M(s) = C(sI - A)^{-1}B + D$ with A Hurwitz, show that

$$\text{ISP} \Rightarrow \text{SPR} \Rightarrow \text{OSP} \Rightarrow \text{PR}$$

b. Prove a counterpart to Lemma 6.2 (Khalil, 3rd ed) with SPR replaced by OSP and (6.14-16) replaced by a convex LMI.

Exercise 3.7 Derive an LMI condition for asymptotic stability in Khalil 7.6. Apply it to 7.6(c) using LMI-lab/IQCbeta/yalmip. How do you take into account the diagonal structure of the nonlinearity?

Exercise 3.8 Find a value of ϵ, the smaller the better, such that

$$\begin{cases} (x-y)^2 < 0.01x^2 \\ (y+0.1x-1.1)^2 < 0.01 \end{cases} \quad \Rightarrow \quad (x-1)^2 + (y-1)^2 < \epsilon$$

Exercise 3.9 For the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ y \end{bmatrix} = \begin{bmatrix} -1+\delta_1 & 2 & 4 & 5 \\ 0 & -2 & 1 & \delta_2 \\ \delta_1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u \end{bmatrix}$$

use μ-tools to compute upper and lower bounds on the worst case gain from u to y as $\delta_1 \in [-0.1, 0.1]$, $\delta_2 \in [-1, 1]$.

Exercise 3.10 Consider a state feedback control systems, where each of the three state measurements has an uncertain time delay:

$$\dot{x}(t) = Ax(t) + B[l_1 x_1(t-\tau_1) + l_2 x_2(t-\tau_2) + l_3 x_3(t-\tau_3)]$$

Suppose that A is stable.
a. Give a frequency domain condition that implies stability for a given triple (τ_1, τ_2, τ_3).
b. Verify that

$$\delta = i\frac{e^{-i\omega\tau} - 1}{e^{-i\omega\tau} + 1}$$

is a real number for all $\omega\tau \in [0, \pi]$. How can this be used together with μ-analysis in verifying closed loop stability for all time delay combinations (τ_1, τ_2, τ_3) with $\tau_k \in [0, 1]$.

Exercise 3.11 Periodic perturbations are common, for example in power systems. Consider the system

$$\dot{x}(t) = Ax(t) - B\cos(\omega t)Cx(t)$$
$$C(sI-A)^{-1}B = \frac{2s}{(s+10)(s^2+0.4s+1)}$$

Find a value of ω_0 such that the system is stable for all $\omega > \omega_0$.

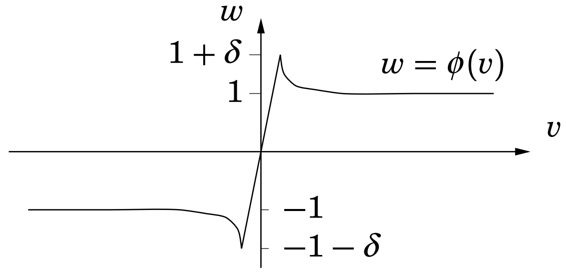

Figure 1 Stiction Nonlinearity

Exercise 3.12 Consider a "stiction" nonlinearity of the form

$$\begin{cases} \phi(v) \in [1, 1+\delta] & \text{if } v > \epsilon(1+\delta) \\ \phi(v) = v/\epsilon & \text{if } v \in [-\epsilon(1+\delta), \epsilon(1+\delta)] \\ \phi(v) \in [-1-\delta, -1] & \text{if } v < -\epsilon(1+\delta) \end{cases}$$

a. Verify that

$$[v(t) - \epsilon w(t)][(1+\delta)w(t) + (h * w)(t)] \;\; \geq \;\; 0$$

provided that $\int_{-\infty}^{\infty} |h(t)| dt \leq 1$. Use this to derive an integral quadratic constraint for the nonlinearity.

b. Verify stability for the closed loop system

$$\ddot{\theta}(t) = u(t) - \phi(\dot{\theta}(t)) \qquad \text{(Servo)}$$

$$u(t) = \; -2\dot{d}(t) - 2d(t) - \int_0^t e^{0.1(\tau - t)} d(\tau) d\tau \qquad \text{(Controller)}$$

$$d(t) = \theta(t) - \theta_{\text{ref}}(t)$$

with stiction level $\delta = 0.1$.