# Lecture 6

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## Modeling of Liquid Slosh for Control

The control problem is to move a container with liquid as fast as possible without too much slosh.

No measurement of the slosh is available for feedback, therefore open loop control is the only possibility.

Open loop control requires a accurate model to be successful.

Tradeoff between design complexity and model complexity.

## Modeling of Liquid Slosh

Navier-Stokes equations for a Newtonian viscous fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \left(\begin{array}{c}\text{Mass}\\ \text{balance}\end{array}\right)$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\nabla^2\mathbf{v} + \rho\mathbf{f} \qquad \left(\begin{array}{c}\text{Force}\\ \text{balance}\end{array}\right)$$

$\rho$ **:** density

$\mathbf{v} = (u,\ v,\ w)$ **:** flow velocity vector field

$p$ **:** pressure

$\lambda$ **:** volume compression factor

$\mu$ **:** viscosity

$\mathbf{f} = (f_x,\ f_y,\ f_z)$ **:** external force vector field

## Modeling of Liquid Slosh

Incompressible ($\lambda = 0$) and inviscid ($\mu = 0$) fluid gives the Euler equations

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{f}$$

If the curl of the flow velocity field is zero ($\nabla \times \mathbf{v} = 0$) there exist a potential $\phi$ such that $\mathbf{v} = \nabla\phi$. If the external force field has a potential $f = -\nabla V$ we get

$$\nabla^2\phi = 0 \qquad \text{(Laplace)}$$

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + V = C(t) \quad \text{(Bernoulli)}$$

where $C(t)$ is an arbitrary function of time.

## Liquid Slosh in a Rectangular Container

A rectangular container with width $a$ and depth $h$ accelerated with horizontal acceleration $u(t)$ gives the following equations

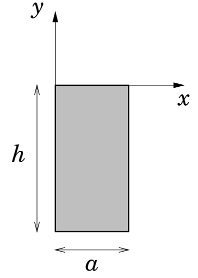
$$\nabla^2\phi = 0 \qquad (1)$$

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gy + u(t)(x - a/2) = C(t) \qquad (2)$$

$$\phi_x(t, 0, y) = 0$$

$$\phi_x(t, a, y) = 0$$

$$\phi_y(t, x, -h) = 0$$



Use separation of variables and define $\phi$ as

$$\phi(t, x, y) = T(t)X(x)Y(y)$$

and insert into $(1)$.

## Calculation of Mode Shapes

This gives

$$T(t)X''(x)Y(y) + T(t)X(x)Y''(y) = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

And we get the following equations for the vertical and horizontal modes

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(a) = 0 \end{cases} \qquad \begin{cases} Y'' - \lambda X = 0 \\ Y'(-h) = 0 \end{cases}$$

$\lambda > 0$ gives the only nontrivial solutions, this gives the horizontal mode

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$
$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

## Calculation of Mode Shapes

Insertion of the boundary conditions gives

$$X'(0) = B\sqrt{\lambda} = 0 \quad \Rightarrow \quad B = 0$$
$$X'(a) = -A\sqrt{\lambda}\sin\sqrt{\lambda}a = 0,\ A \neq 0 \quad \Rightarrow \quad \lambda = \frac{n^2\pi^2}{a^2},\ n = 1, 2, \ldots$$

The vertical mode becomes

$$Y(y) = A\cosh\frac{n\pi}{a}y + B\sinh\frac{n\pi}{a}y$$
$$Y'(y) = A\frac{n\pi}{a}\sinh\frac{n\pi}{a}y + B\frac{n\pi}{a}\cosh\frac{n\pi}{a}y$$

Insertion of the boundary conditions gives

$$Y'(-h) = -A\frac{n\pi}{a}\sinh\frac{n\pi}{a}h + B\frac{n\pi}{a}\cosh\frac{n\pi}{a}h = 0 \quad \Rightarrow \quad A = \frac{B\cosh\frac{n\pi}{a}h}{\sinh\frac{n\pi}{a}h}$$

## Calculation of Mode Shapes

This gives the following mode shapes

$$\begin{aligned} X(x) &= A\cos\frac{n\pi}{a}x \\ Y(y) &= \frac{B\cosh\frac{n\pi}{a}h}{\sinh\frac{n\pi}{a}h}\cosh\frac{n\pi}{a}(y+h) \end{aligned} \qquad n = 1, 2, \ldots$$

which gives the potential

$$\phi(t, x, y) = \sum_{n=1}^{\infty}\cos\frac{n\pi}{a}x\cosh\frac{n\pi}{a}(y+h)T_n(t)$$

where all constants are lumped together in $T_n(t)$.

## Calculation of Acceleration Response

Define $S(t, x)$ as the surface elevation above the point $(x, 0)$.

The pressure on the surface is equal to the pressure in the medium above the surface, set $C(t) = \frac{p(t,x,S(t,x))}{\rho}$.

Evaluation of $(2)$ on the free surface gives

$$\frac{\partial\phi(t, x, S(t, x))}{\partial t} + gS(t, x) + u(t)(x - a/2) = 0 \qquad (3)$$

Differentiation with respect to time gives

$$\frac{\partial^2\phi(t, x, S(t, x))}{\partial t^2} + \frac{\partial^2\phi(t, x, S(t, x))}{\partial t\partial y}\dot{S}(t, x) + g\dot{S}(t, x) + \dot{u}(t)(x - a/2) = 0$$

## Calculation of Acceleration Response

Assuming that $S$ is small, neglecting the nonlinear term and $\dot{S}(t,x) = \frac{\partial\phi(t,x,S(t,x))}{\partial y}$ gives

$$\frac{\partial^2\phi(t,x,0)}{\partial t^2} + g\frac{\partial\phi(t,x,0)}{\partial y} = -\dot{u}(t)(x-a/2)$$

Insertion of $\phi(t,x,0)$ gives

$$\sum_{n=1}^{\infty}\left(\cosh\frac{n\pi h}{a}T_n''(t) + \frac{n\pi g}{a}\sinh\frac{n\pi h}{a}T_n(t)\right)\cos\frac{n\pi}{a}x = -\dot{u}(t)(x-a/2)$$

Fourier expansion of RHS gives

$$\sum_{n=1}^{\infty}\left(\cosh\frac{n\pi h}{a}T_n''(t) + \frac{n\pi g}{a}\sinh\frac{n\pi h}{a}T_n(t)\right)\cos\frac{n\pi}{a}x = \\ -2a\dot{u}(t)\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi^2}\cos\frac{n\pi}{a}x$$

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## Calculation of Acceleration Response

The uniqueness of the Fourier expansion gives the following differential equations for $T_n(t)$

$$T_n''(t) + \omega_n^2 T_n(t) = \begin{cases} 0, & n \text{ even} \\ b_n\dot{u}(t), & n \text{ odd} \end{cases}$$

with

$$\omega_n = \sqrt{\frac{n\pi g}{a}\tanh\frac{n\pi h}{a}} \qquad b_n = \frac{4a}{n^2\pi^2\cosh\frac{n\pi h}{a}}$$

Note that the applied acceleration only excites the odd modes.

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## Calculation of the Surface Elevation

Evaluation of $(3)$ in $x_m$ gives

$$S(t,x_m) = -\frac{1}{g}\left(\frac{\partial\phi(t,x_m,0)}{\partial t} + u(t)(x_m-a/2)\right)$$

Insertion of $\phi(t,x_m,0)$ gives

$$S(t,x_m) = -\frac{1}{g}\left(\sum_{n=1,\ n\ \text{odd}}^{\infty} c_n(x_m)T'(t) + u(t)(x_m-a/2)\right)$$

with

$$c_n(x_m) = \cos\frac{n\pi x_m}{a}\cosh\frac{n\pi h}{a}, \quad n = 1,3,\ldots$$

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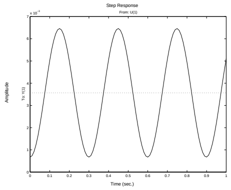
## The model

A model describing the surface elevation in $x_m$ is now given by

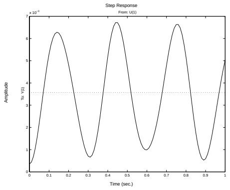
$$S(t,x_m) = -\frac{1}{g}\left(\sum_{n=1,\ n\ \text{odd}}^{\infty}\frac{b_n c_n(x_m)p^2}{p^2+w_n^2} + x_m - \frac{a}{2}\right)u(t)$$

where $p$ is the differential operator $\frac{d}{dt}$.

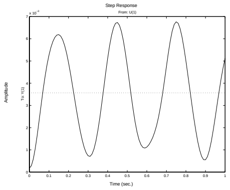
Step response for different number of modes



n=1



n=3

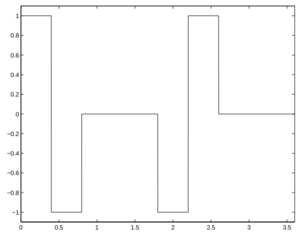


n=7

## Comparison with Real Slosh

Applied acceleration

$$u(t) = \begin{cases} u_{max} & 0 \le t < 0.4 \\ -u_{max} & 0.4 \le t < 0.8 \\ 0 & 0.8 \le t < 1.8 \\ -u_{max} & 1.8 \le t < 2.2 \\ u_{max} & 2.2 \le t < 2.6 \\ 0 & 2.6 \le t < 3.6 \end{cases}$$



Experiments are performed with $u_{max} = [0.25,\ 1,\ 3]$ m/s$^2$ on a container with $h = 0.2$ m and $a = 0.07$ m.

The slosh is measured on the left side of the container with an infrared laser displacement sensor.

Simulated slosh close to the real slosh for small surface elevation oscillation amplitudes.