MOTION CONTROL OF OPEN CONTAINERS WITH SLOSH CONSTRAINTS

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Abstract: Movement of open containers containing liquid is considered. A simple linear slosh model is presented that captures the main features of the slosh phenomena. The open-loop acceleration trajectory is calculated using optimal control techniques. Both minimum-time and minimum-energy problems are solved with various constraints on the control and state trajectories. The calculated acceleration profiles are evaluated using experiments showing better performance than previous controllers.

Keywords: Slosh modeling, Optimal control, Motion control, Minimum time, Minimum energy, Servo system, Experimental evaluation

1. INTRODUCTION

This paper considers movement of open containers containing liquid. This is a common operation in the packaging industry. Improved movement methods are directly reflected in the production rate. Increased production rate gives a lower packaging cost and higher profit.

The operation of a packaging machine can be divided into three independent sub tasks: folding, filling and sealing. These tasks are performed simultaneously on three different packages. A schematic picture of a packaging machine is shown in Fig. 1.

The folded package is placed in a holder that carries the package through the machine. The movement of the package is performed stepwise, the number of steps between the different sub tasks depend on the machine type. The same movement is applied in every step on all packages. The time it takes to produce a package is determined by the filling time, which is the slowest of the sub tasks, and the time it takes to move the package one step.

The package contains liquid when it is moved between the filling and the sealing stations. The movement induces liquid motion in the package. This is what we refer to as slosh. The amount of slosh depends on how the package is accelerated and the properties of the liquid. There are large differences between skim milk and yoghurt.

If there is too much slosh the liquid might splash on the surfaces that should be be glued. This can result in packages that are not properly sealed and possibly not airtight. If the

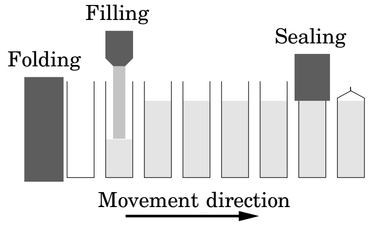

Fig. 1. Schematic picture of the packaging machine.

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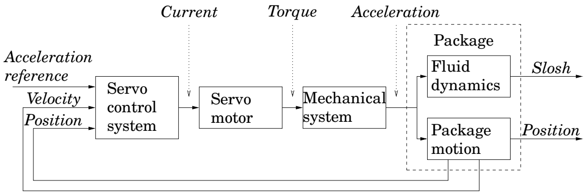

Fig. 2. Block diagram of the motion control system.

package is not airtight the storage time is decreased considerably. This is particularly critical for aseptic packages that are supposed to have a very long storage time.

The only way to increase the production capacity of the machine is by decreasing either the filling time or the movement time or both. This paper considers the problem of decreasing the movement time.

The movement of the packages in the machine is controlled by a servo system which controls the position of the packages. A block diagram of the motion control system is shown in Fig. 2. The reference is specified as an acceleration profile which is integrated twice to generate a position reference. The only measurements that are available to the servo system are the position and velocity of the package. Therefore, the only way to control the slosh is through the acceleration reference.

The problem is to calculate an open-loop acceleration profile that moves the package one step on minimum time with acceptable slosh. During the movement the slosh has to be bounded below a certain level. The same acceleration profile is applied at each step. Therefore, the acceleration must be such that the slosh constraint is not violated when the acceleration profile is repeated. One way to achieve this is to ensure that the slosh is in the same state at the beginning of each movement step. The natural choice of initial state of the slosh is that the liquid is at rest, since this is the state after the package has been filled.

An industrial testbed has been used where a carriage is mounted on a belt driven by a servo system similar to the one in the packaging machine. The container and an infrared laser displacement sensor used to measure the surface elevation is mounted on the carriage. A schematic picture of the setup is shown in Fig. 3.

The problem is solved by first deriving a model of the slosh and then applying optimal control techniques to calculate the acceleration profile for the container. The control performance is then evaluated by experiments. A more detailed description of the problem and solutions to the movement problem when the package is moved only one step is found in Grundelius (1998).

2. MODELING

Modeling of the sloshing behavior of liquids in externally excited containers has been studied in several applications, e.g. fuel slosh in aerospace applications, vehicle and ship dynamics, earthquake engineering and movement of containers. The choice of slosh model is non-trivial. The most advanced detailed model is described by a set of three dimensional nonlinear partial differential equations (PDE). Such models are hard to use in controller design, but can be useful for simulation and analysis. The nonlinear effects are important for very rapid movements when the surface elevation is large.

Numerical solutions to the 2D problem using the finite element method (FEM) is given in Armenio and La Rocca (1996) and Kelkar and Patankar (1997), and using the boundary element method (BEM) in Romero and Ingber (1995). Analytical solutions to the slosh problem can be obtained by approximations and is presented in Coulson (1955) and Venugopal and Bernstein (1996). These solutions are only valid for oscillation amplitudes much smaller than the wave length.

Fig. 4 shows the results of an impulse response experiment. In the figure two nonlinear phenomena can be observed: the oscillation is asymmetric and the oscillation frequency is slightly amplitude dependent (the oscillation frequency increases with decreasing amplitude).

A simple linear model with four states is used which captures most of the behavior. The slosh model is given by the state space model

$$\dot{x} = \underbrace{\begin{bmatrix} -2\zeta\omega & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} a\omega/2g \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{B} u \qquad (1)$$

where x_2 is the surface elevation in meters, x_1 is the time derivative of the surface elevation divided by ω, x_3 is the container velocity and x_4 the container position, see Fig. 5.

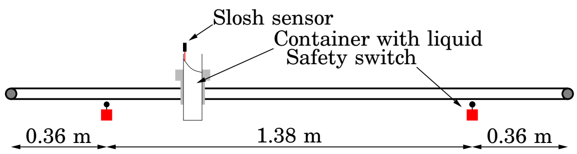

Fig. 3. Schematic picture of the testbed. A liquid container and an infrared laser displacement sensor is mounted on the carriage.

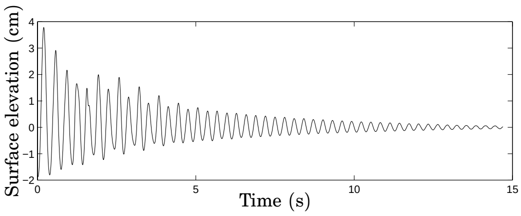

Fig. 4. An impulse is made in the acceleration and the surface elevation is measured in a point close to the container wall. Two nonlinear phenomena can be noticed: asymmetric oscillation and amplitude dependent oscillation frequency.

For a rectangular container with liquid depth h and width a the oscillation frequency of the first harmonic is given by the expression below, see Coulson (1955) and Venugopal and Bernstein (1996).

$$\omega = \sqrt{\frac{g\pi}{a}\tanh\frac{h\pi}{a}} \tag{2}$$

Throughout this paper a container with $h = 0.2$ m and $a = 0.07$ m is studied, which gives the theoretical value $\omega = 21.0$ rad/s.

For later use the following definition is introduced.

Definition 1. If $s(t)$ denotes the surface elevation and $\dot{s}(t)$ its time derivative the stored internal slosh $s_i(t)$ is defined as

$$s_i(t) = \sqrt{s^2(t) + \left(\frac{\dot{s}(t)}{w}\right)^2}$$

The stored internal slosh represents the slosh amplitude of the remaining oscillation if the acceleration is set to zero after time t, if the damping $\zeta = 0$.

3. CALCULATION OF ACCELERATION PROFILES

The industrial, practical solution to the movement problem has been to use ad-hoc guessing to determine the structure of the acceleration profile. The development engineers then use

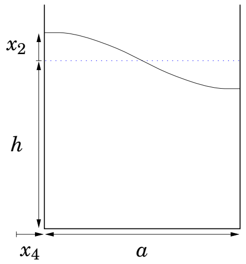

Fig. 5. Illustration of the container.

experiments and experience to tune the parameters. This procedure is very time consuming and therefore it is of great interest to develop methods to calculate a good acceleration profile.

A solution is presented in Feddema et al. (1997). The approach is based on a second order linear slosh model, described by a transfer function $G(s)$. The acceleration profile is obtained by filtering a bang-bang signal trough the filter $G_d(s)G^{-1}(s)$, where $G_d(s)$ is the transfer function of the desired third order critically damped slosh response.

In Dietze and Schmidt (1997) the problem is solved using optimal control techniques and a linear second order slosh model. In the optimization the control signal is discretized and the cost function

$$\frac{\alpha^2}{2}[s^2(T) + \dot{s}^2(T)] + \frac{\beta^2}{2}\int_0^T u^2(T)\, dt$$

is minimized for different values of α and β and with various constraints on $u(t)$ and $s(t)$. The problem with $\alpha = 0$, $|u(t)| \le u_{max}$ and no constraint on the surface elevation $s(t)$ is solved analytically.

In Dubois et al. (1998) the problem is approached using flat systems.

In this paper optimal control techniques are applied to calculate the acceleration profile. Both the minimum-time and minimum-energy problem have been solved subject to various constraints both numerically and analytically.

3.1 *Constraints*

The following constraints have been used when solving the optimal control problems:

(1) Acceleration: $|u(t)| \le u_{max} = 9.81$ m
(2) Slosh: $|x_2(t)| \le s_{max} = 0.035$ m
(3) Initial state: $x(0) = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$
(4) Terminal state: $x(T) = \begin{bmatrix}0 & 0 & 0 & L\end{bmatrix}^T$

The movement distance is $L = 0.2$ m.

3.2 *Minimum-time problem*

To find an acceleration profile that minimizes the movement time, the following cost function is minimized subject to Constraint 1-4 and the slosh model.

$$J = \int_0^T 1\, dt$$

The optimization problem is solved numerically using a Matlab Toolbox called RIOTS (Recursive Integration Optimal Trajectory Solver), see

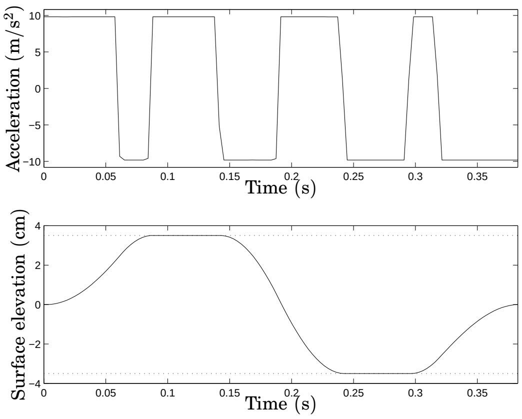

Fig. 6. Simulation of the minimum-time acceleration profile, the minimum time is $T = 0.3826$ s.

Schwartz and Polak (1996). The minimum-time solution is shown in Fig. 6.

Practical evaluations of the minimum-time strategy in the experimental setup have shown that it only works for small values of s_{max} and u_{max}. Typical results of the experiments are shown in Fig. 7. The figure shows that for a small value of u_{max} the slosh corresponds to the calculated slosh shown in Fig. 6 except for a high frequency oscillation, but as u_{max} is increased the performance degrades.

Insight into why the performance degrades is obtained by studying the stored internal slosh shown in Fig. 8. The figure shows that in the middle of the movement s_i is first increased and then decreased. This pumping of energy in and out of the system requires a very accurate model to be successful. From Fig. 7 it can also be seen that the largest deviations come in the middle of the movement.

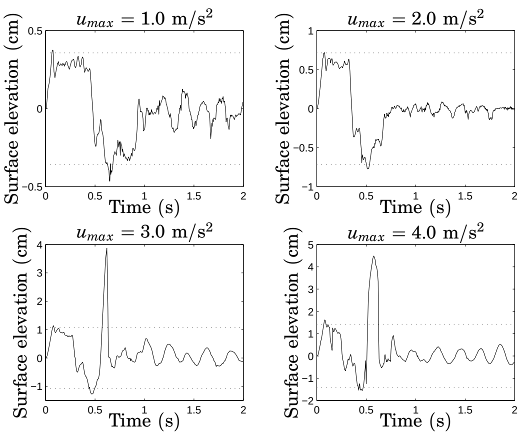

Fig. 7. Data from experiments with the minimum-time acceleration profile for varying values of u_{max}. There is bad agreement with simulations for $u_{max} > 2$ m/s^2. The dotted lines show $\pm s_{max} = \pm \frac{a}{2g} u_{max}$. This is a motivation for the minimum-energy approach.

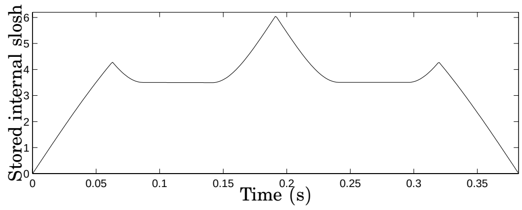

Fig. 8. The stored internal slosh as a function of time for the minimum-time acceleration profile. The big peak in the middle shows that energy is pumped in and out of the system, this requires an accurate model to be successful. This might be an explanation of the problems shown in Fig. 7.

3.3 *Minimum-energy problem*

One way of making the acceleration profile smoother is to solve the minimum-energy problem instead. Therefore, the cost function

$$J = \int_0^{T_{opt}+\Delta} u^2(t)\, dt$$

is minimized subject to Constraint 1-4 and the slosh model, where T_{opt} is the time from the solution of the minimum-time problem and Δ is the extra time we can allow for the movement.

The problem is solved numerically and the result is shown in Fig. 9 for different values of Δ. The figure shows that by increasing the movement time slightly we can make the acceleration profile much smoother.

Analytical solutions to the minimum-energy problem can actually be easily obtained with the following modifications of the constraints:

- Remove the control constraint 1
- Replace the slosh inequality constraint 2 with a quadratic penalty on the slosh
- Replace the terminal state inequality constraint 4a with a quadratic penalty on the terminal state

This gives the following cost function and constraints

$$\begin{aligned} J &= \frac{1}{2} \int_0^T x^T(t) Q x(t) + R u^2(t)\, dt \\ x(0) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ x(T) &= \begin{bmatrix} 0 & 0 & 0 & L \end{bmatrix}^T \end{aligned} \tag{3}$$

The control $u(t)$ that minimizes J is obtained by simultaneously solving the system equation (1) and the Euler-Lagrange equations, see Bryson and Ho (1975)

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} \tag{4}$$

$$\frac{\partial H}{\partial u} = 0 \tag{5}$$

with the Hamiltonian

$$H = \frac{1}{2}(x^T Q x + R u^2) + \lambda^T(Ax + Bu)$$

Equation (1), (4) and (5) give

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -\frac{1}{R} B B^T \\ -Q & -A^T \end{bmatrix}}_{M} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad (6)$$

and

$$u = -\frac{1}{R} B^T \lambda \quad (7)$$

The solutions to (6) can be written as

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} \quad (8)$$

where $\Phi(t) = e^{Mt}$.

Insertion of $x(0)$ in (8) gives

$$x(t) = \Phi_{1:4,5:8}(t)\lambda(0) \quad (9)$$

$$\lambda(t) = \Phi_{5:8,5:8}(t)\lambda(0) \quad (10)$$

where $\Phi_{1:4,5:8}$ means the sub matrix with row 1 to 4 and column 5 to 8 of Φ. Evaluation of (9) at time T gives

$$\lambda(0) = [\Phi_{1:4,5:8}(T)]^{-1} x(T) \quad (11)$$

The optimal control $u(t)$ is hence

$$u(t) = -\frac{1}{R} B^T \Phi_{5:8,5:8}(t) [\Phi_{1:4,5:8}(T)]^{-1} x(T)$$

For the case when there is no penalty on the state trajectory ($Q = 0$, $R = 1$) $\Phi(t)$ is

$$\Phi(t) = \begin{bmatrix} e^{At} & \Phi_{1:4,5:8}(t) \\ 0 & e^{-A^T t} \end{bmatrix}$$

and with $\zeta = 0$

$$B^T \Phi^T_{5:8,5:8}(t) = \begin{bmatrix} \frac{a\omega}{2g} \cos \omega t & -\frac{a\omega}{2g} \sin \omega t & 1 & -t \end{bmatrix}$$

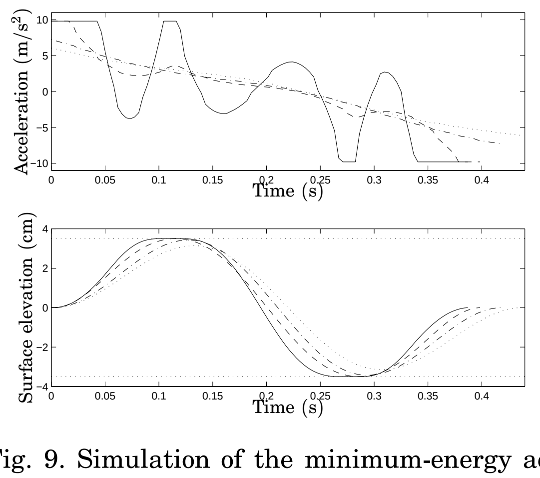

Fig. 9. Simulation of the minimum-energy acceleration profiles for different values of Δ, $\Delta = 0.02T_{opt}$ (solid), $\Delta = 0.05T_{opt}$ (dashed), $\Delta = 0.10T_{opt}$ (dash dotted) and $\Delta = 0.15T_{opt}$ (dotted). The acceleration profile is much smoother already for a small increase of the movement time T.

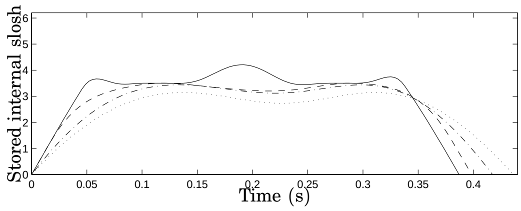

Fig. 10. The stored internal slosh as a function of time for the minimum-energy acceleration profile for movement in several steps for different values of Δ, $\Delta = 0.02T_{opt}$ (solid), $\Delta = 0.05T_{opt}$ (dashed), $\Delta = 0.10T_{opt}$ (dash dotted) and $\Delta = 0.15T_{opt}$ (dotted). Compare with Fig. 8.

this gives the optimal solution

$$u(t) = -\frac{a\omega}{2g}(\lambda_1(0)\cos\omega t - \lambda_2(0)\sin\omega t) + \lambda_3(0) - \lambda_4(0)t \quad (12)$$

where $\lambda(0)$ is given by (11).

The control strategy in (12) is evaluated for the movement in several steps. The container is moved five times and the movement time is 0.46 s and the filling time between the movements is 0.44 s. If the nominal value of $\omega = 21.0$ given by (2) is used when calculating the acceleration profile we do not get the expected performance. A way of defining performance is to study the remaining oscillation after one movement is performed.

Definition 2. Define the residual slosh $r(t)$ as

$$r(t) = s(t + T) \ \forall t \geq 0$$

where $s(t)$ is the surface elevation and T is the movement time. The performance measure is defined as

$$R = \sqrt{\int_0^\infty r^2(t)\, dt}$$

Fig. 11 shows the performance measure R for different values of ω. A minimum is observed for $\omega = 19.1$. The surface elevation for acceleration profiles designed for $\omega = 21.0$ and $\omega = 19.1$ are compared in Fig. 12. There is a better agreement for $\omega = 19.1$. Note that there is

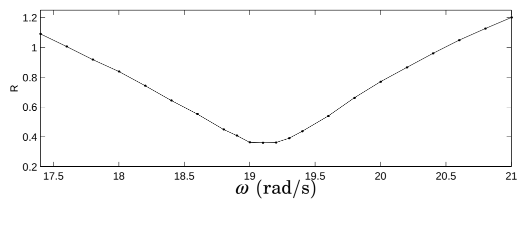

Fig. 11. The performance measure R for different values of ω. A minimum is observed at $\omega = 19.1$, the theoretical value for small amplitudes is $\omega = 21.0$.

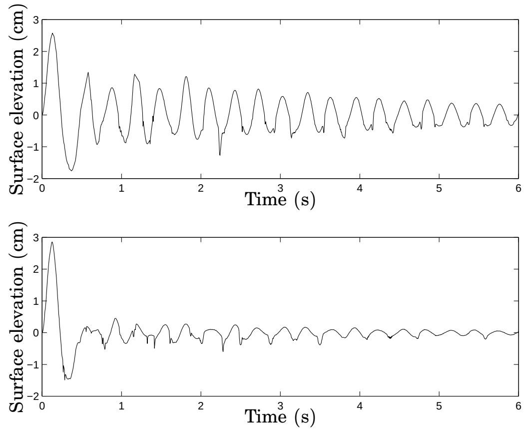

Fig. 12. Experimental data: surface elevation for $\omega = 21.0$ (upper) and $\omega = 19.1$ (lower).

still some residual slosh for $t > 0.46$ s, due to oscillations in the direction perpendicular to the movement. Such oscillations are hard to model.

The surface elevation when five consecutive movements are performed is shown in Fig. 13. The figure shows that the acceleration profile performs very well. Since the residual slosh is small the performance is independent of the time between the movements (i.e. the filling time). Therefore the same acceleration profile works even if the filling time is changed.

4. CONCLUSIONS

An industrially relevant problem has been described where an open container with liquid should be moved quickly without slosh. A linear fourth order model has been verified using real data from an industrial testbed. The problem has been solved using techniques from optimal control. Experiments have shown that the minimum-time strategy is very sensitive to model uncertainty. Robustness is achieved by increasing the movement time and minimizing the energy. The calculated controllers have been implemented and verified to give better performance than previous controllers. Future work includes modeling of the nonlinear effects encountered and application of the method on other liquids and package geometries.

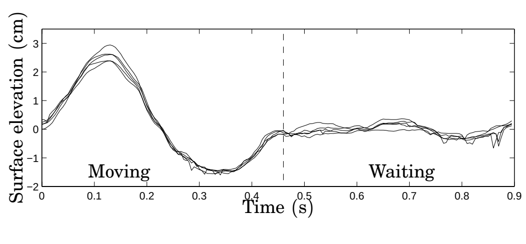

Fig. 13. Experiment when the container is moved five times showing the surface elevation, the movement cycles are plotted upon each other.

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