Riccati Equations and Inequalities in Robust Control

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Riccati (in)equalities are not only used to express optimality conditions (Libertzon chapter 6). They also play a central role in finding robust controllers that can ensure a bound on the ”worst-case” disturbance gain: $H_\infty$ - synthesis.

This presentation will cover the material in Libertzon section 7.3.
Present the result (theorem?) that connects a Riccati equation to the $H_\infty$ norm of a system.
How this can be used in closed loop synthesis.
How to practically solve Riccati inequalities using LMI solvers.
Solve a Riccati equation and ensure a bound on the $u$ to $y$ $L_2$ gain.

Theorem

Given the system: $\dot{x} = Ax + Bu$, $y = Cx$

The two following statements are equivalent:

$\exists$ $P$ that fulfills 1,2:

1) $P \geq 0$
2) $PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P \leq 0$

$$\sup_{u \in L_2 \setminus \{0\}} \frac{\sqrt{\int_0^\infty |y(t)|^2 dt}}{\sqrt{\int_0^\infty |u(t)|^2 dt}} = \sup_{\omega \in \mathbb{R}} |G(j\omega)| \leq \gamma.$$
The Main Theorem

Proof Explanation

Proof (Not Complete).

Solve the following optimal control problem:

Minimize:  \( J(u) = \int_0^\infty \left( \gamma u^T(t)u(t) - \frac{1}{\gamma} y^T(t)y(t) \right) dt \)

Subject to:  \( \dot{x} = Ax + Bu, \ y = Cx. \)

It is straightforward to show that if there exists a \( P \) fulfilling 1,2 (equality) and if \( A + \frac{1}{\gamma} BB^T P \) is Hurwitz.

Then the optimal cost is given by:

\( V(x_0) = -x_0^T Px_0 \)

obtained by:

\( u^*(t) = \frac{1}{\gamma} B^T Px^*(t). \)
Proof (Cont.)

The optimal control result implies that:

\[-x_0^T P x_0 \leq \int_0^\infty \left( \gamma u^T(t) u(t) - \frac{1}{\gamma} y^T(t) y(t) \right) dt\]

Then if \( x_0 = 0 \implies \sup_{u \in L_2 \setminus \{0\}} \frac{\sqrt{\int_0^\infty |y(t)|^2 dt}}{\sqrt{\int_0^\infty |u(t)|^2 dt}} = \sup_{\omega \in \mathbb{R}} |G(j\omega)| \leq \gamma.\]

Then it can be shown that the Hurwitz condition could be relaxed and the Riccati equality could be relaxed to an inequality.
This result can be used to find robust controllers.

Consider the system:

\[
\dot{x} = Ax + Bu + D\omega \\
y = Cx \\
z = Ex
\]

With the assumptions that:

\( C = I \) (all the states are known.)

\( u = Kx \) (Static state feedback control.)
The $H_\infty$ control problem

The design problem of finding $K$ so that:

1. $A + BK$ is Hurwitz.
2. The $H_\infty$ norm of the closed loop system from $\omega$ to $z$ should be less than a predetermined value $\gamma$.

Since we have the closed loop system:

$$\dot{x} = (A + BK)x + D\omega$$
$$z = Ex$$

We can use the previously presented theorem to fulfill 2 by finding $P \geq 0$ such that:

$$PA_{cl} + A_{cl}^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP \leq 0.$$

The $H_\infty$ control problem

To also encode the stability criterion into the Riccati equation, one can introduce $Q > 0$ and $\epsilon > 0$ and instead solve the equality:

$$PA_{cl} + A_{cl}^T P + \frac{1}{\gamma} E^T E + \frac{1}{\gamma} PDD^T P + \epsilon Q = 0.$$ 

Since this implies that:

$$PA_{cl} + A_{cl}^T P < 0, \quad \implies \quad A_{cl} \text{ is Hurwitz.}$$
The $H_\infty$ control problem

Now we have an equation with $K$ and $P$ unknown:

$$PA_{cl} + A_{cl}^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP + \epsilon Q = 0.$$ 

To make the process of finding $K$ and $P$ easier one can introduce a tuning matrix $R > 0$, and instead solve:

$$PA + A^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP + \epsilon Q - \frac{1}{\epsilon}PBR^{-1}B^TP = 0.$$ 

and then let

$$K = -\frac{1}{2\epsilon}R^{-1}B^TP.$$
If the states are not measured..

There is a result saying that a dynamic controller satisfying 1 and 2 can be found by solving the two Riccati equalities:

\[ P_1 A + A^T P_1 + \frac{1}{\gamma} E^T E + \frac{1}{\gamma} P_1 D D^T P_1 + \epsilon Q - \frac{1}{\epsilon} P_1 B R^{-1} B^T P_1 = 0. \]

\[ P_2 A + A^T P_2 + \frac{1}{\gamma} D^T D + \frac{1}{\gamma} P_2 E E^T P_2 + \epsilon Q - \frac{1}{\epsilon} P_2 C R^{-1} C^T P_2 = 0. \]

where the largest singular value of \( P_1 P_2 \) \( \leq \gamma^2 \)

The found controller can be interpreted as a state feedback law combined with an observer.
Solving a Riccati inequality

To ensure that the $L_2$ gain is less than or equal to $\gamma$:

$$P = P^T \geq 0$$

$$PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P \leq 0$$
Convert it to an LMI (Linear Matrix Inequality)

\[ PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P \leq 0 \] (Unfortunately quadratic in \( P \))

\[ \Leftrightarrow \]

\[ \begin{pmatrix} PA + A^T P + \frac{1}{\gamma} C^T C & \sqrt{\frac{1}{\gamma}} PB \\ \sqrt{\frac{1}{\gamma}} B^T P & -I \end{pmatrix} \leq 0 \] (The Schur complement)

This LMI can be solved numerically, using e.g. the Robust Control Toolbox in Matlab.
Solving an example

Consider the system

\[ G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \]

where \( \omega_0 = 1 \) and \( \zeta = 0.3 \)
Solving an example

1) Find a state-space representation of the system

2) Formulate the Linear Matrix Inequality in Matlab

\[
\begin{pmatrix}
PA + A^T P + \frac{1}{\gamma} C^T C & \sqrt{\frac{1}{\gamma}} PB \\
\sqrt{\frac{1}{\gamma}} B^T P & -I
\end{pmatrix} \leq 0
\]

3) Solve the inequality for \( P \) using feasp

4) Verify the solution
Solving an example using feasp

Solver for LMI feasibility problems $L(x) < R(x)$ This solver minimizes $t$
subject to $L(x) < R(x) + t*I$
The best value of $t$ should be negative for feasibility

$$P = \begin{pmatrix} 1.7697 & 0.3246 \\ 0.3246 & 1.7699 \end{pmatrix}$$
Success!

Sometimes, however:
Result: best value of $t$: 0.407204
These LMI constraints were found infeasible
The End