### Exact Feedback Linearization

**Idea:**

Find state feedback $u = u(x,v)$ so that the nonlinear system

$$\dot{x} = f(x) + g(x)u$$

turns into the linear system

$$\dot{x} = Ax + Bv$$

and then apply linear control design method.

Introduce new control variable $v$ and let

$$u = M(\theta)v + C(\theta, \dot{\theta}) + G(\theta)$$

This gives the closed-loop system:

$$\ddot{\theta} + K_d \dot{\theta} + K_p \theta = K_p \theta_{ref}$$

Hence, $u = M(\theta)v + C(\theta, \dot{\theta}) + G(\theta) = u(x)$ with $x = (\theta, \dot{\theta})^T$

### Computed Torque

The computed torque

(also known as “Exact linearization”, “dynamic inversion”, etc.)

$$u = M(\theta)v + C(\theta, \dot{\theta}) + G(\theta)$$

where $d$ is the viscous damping. The control $u = \tau$ is the applied torque

Design state feedback controller $u = u(x)$ with $x = (\theta, \dot{\theta})^T$

### Lyapunov-Based Control Design Methods

Select Lyapunov function $V(x)$ for stability verification

Find state feedback $u = u(x)$ that makes $V$ decreasing

Method depends on structure of $f$

Examples are energy shaping as in Lab 2 and, e.g., Back-stepping control design, which require certain $f$ discussed later.
Use Lyapunov-based design for swing-up control.

Example of Lyapunov-based design

Consider the nonlinear system
\begin{align}
\dot{x}_1 &= -3x_1 + 2x_1 x_2^2 + u \\
\dot{x}_2 &= -x_2^2 - x_2,
\end{align}

where we chose
\[ u = -x_2 - x_2^2 x_1 \]

Find a globally asymptotically stabilizing control law \( u = u(x) \).

**Attempt 1:** Try the standard Lyapunov function candidate
\[ V(x_1, x_2) = \frac{1}{2} (x_1^2 + x_2^2), \]

which is radially unbounded. \( V(0, 0) = 0 \), and \( V(x_1, x_2) > 0 \) \( \forall (x_1, x_2) \neq (0, 0) \).

\[ \dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = x_2^2 x_1 + u x_2 = x_2^2 x_1 + u \]

However \( \dot{V} = 0 \) as soon as \( x_2 = 0 \) (Note: \( x_1 \) could be anything).

According to LaSalle’s theorem the set \( E = \{ x | V = 0 \} = \{ x_1, 0 \} \) \( \forall x_1 \).

What is the largest invariant subset \( M \subseteq E \)?

Plugging in the control law \( u = -x_2 - x_2^2 x_1 \), we get
\[ \dot{x}_1 = x_2^2 x_1 + u x_2 = x_2^2 x_1 + u \]

Observe that if we start anywhere on the line \( \{ (x_1, 0) \} \) we will stay in the same point as both \( \dot{x}_1 = 0 \) and \( \dot{x}_2 = 0 \), thus \( M = E \) and we will not converge to the origin, but get stuck on the line \( x_2 = 0 \).

**Attempt 2:**
\begin{align}
\dot{x}_1 &= x_2^2 \\
\dot{x}_2 &= u
\end{align}

Try the Lyapunov function candidate
\[ V(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{4} x_2^4, \]

which satisfies
\[ \dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = x_1^2 x_1 + u x_2 = -x_2^4 \leq 0 \]

if we use \( u = -x_1 - x_2 \).
Thus, the largest invariant set \( M \) end up in \( \{x_1, 0\} \). According to the Invariant Set Theorem (LaSalle) all solutions will end up in \( M \) and so the origin is GAS.

Introduce \( \tilde{x} = x - \hat{x} \), \( \hat{a} = a - \tilde{a} \), \( \hat{b} = b - \tilde{b} \). Want to design adaptation law so that \( \tilde{x} \to 0 \)

Example

\[
\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u = Ax + Bu
\]

\( u = -\text{sgn}(x) = -\text{sgn}(Cx) \)

which means that

\[
\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}
\]

Determine the sliding set and the sliding dynamics.


\[ \dot{x}_1 = -x_2 + u = -x_2 - \text{sgn}(x_2) \]
\[ \dot{x}_2 = x_1 - x_2 + u = x_1 - x_2 - \text{sgn}(x_2) \]

\[ \sigma(x) = 0 \quad f^+ = \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} \quad f^- = \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix} \]

\[ \sigma(x) = 0 \quad x_2 = 0 \]
\[ \frac{\partial \sigma}{\partial x} f^+ < 0 \quad x_1 - x_2 - 1 < 0 \]
\[ \frac{\partial \sigma}{\partial x} f^- > 0 \quad x_1 - x_2 + 1 > 0 \]

We thus have the sliding set \{\(-1 < x_1 < 1, x_2 = 0\)\}

### Design of Sliding Mode Controller

**Idea:** Design a control law that forces the state to \(\sigma(x) = 0\).

Choose \(\sigma(x)\) such that the sliding mode tends to the origin.

Assume system has form

\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(x) + g_1(x)v \\ f_2(x) + g_2(x)v \\ \vdots \\ f_n(x) + g_n(x)v \end{bmatrix} \]

Choose control law

\[ u = \frac{p^T f(x)}{p^T g(x)} - \theta \frac{p^T g(x)}{p^T g(x)} \text{sgn}(x), \]

where \(\theta > 0\) is a design parameter, \(\sigma(x) = p^T x\), and \(p^T = [p_1 \ldots p_n]\) represents a stable polynomial.

### Sliding Mode Control gives Closed-Loop Stability

The sliding dynamics are given by

\[ \dot{x} = f(x) + (1 - \alpha) f^- \]
\[ 0 = \frac{\partial \sigma}{\partial x} f^-(x) + \alpha(x) \frac{\partial \sigma}{\partial x} [f^+(x) - f^-(x)] \]

On the sliding set \{\(-1 < x_1 < 1, x_2 = 0\)\}, this gives

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \alpha \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix} \]
\[ 0 = x_1 - x_2 + 1 - 2\alpha \]

Eliminating \(\alpha\) gives

\[ \dot{x}_1 = -x_1 \quad \dot{x}_2 = 0 \]

Hence, any initial condition the sliding set will give exponential convergence to \(x_1 = x_2 = 0\).
Time to Switch

Consider an initial point \( x \) such that \( \sigma_0 = \sigma(x) > 0 \). Then
\[
\dot{\sigma} = p^T \dot{x} = p^T (f + gu) = -\mu \text{sgn}(\sigma) = -\mu
\]
Hence, the time to the first switch is
\[
t_s = \frac{\sigma_0}{\mu} < \infty
\]
Note that \( t_s \to 0 \) as \( \mu \to \infty \).

Example—Sliding Mode Controller

Design state-feedback controller for
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x
\end{align*}
\]
Choose \( p_1 s + p_2 = s + 1 \) so that \( \sigma(x) = x_1 + x_2 \). The controller is given by
\[
u = -p^T A x - \mu p^T B \text{sgn}(\sigma(x)) = -2x_1 - \mu \text{sgn}(x_1 + x_2)
\]

Phase Portrait

Simulation with \( \mu = 0.5 \). Note the sliding set is in \( \sigma(x) = x_1 + x_2 \).

Time Plots

Initial condition \( x(0) = [1.5, 0]^T \).
Simulation agrees well with time to switch \( t_s = \frac{\sigma_0}{\mu} = 3 \) and sliding dynamics \( \dot{y} = -y \).

The Sliding Mode Controller is Robust

Assume that only a model \( \dot{x} = \hat{f}(x) + \hat{g}(x)u \) of the true system \( \dot{x} = f(x) + g(x)u \) is known. Still, however,
\[
\dot{V} = \sigma(x) \left[ \frac{p^T (f \hat{g}^T - \hat{f} g^T) p}{p^T g} - \mu \frac{p^T g}{p^T g} \text{sgn}(x) \right] < 0
\]
if \( \text{sgn}(p^T g) = \text{sgn}(p^T \hat{g}) \) and \( \mu > 0 \) is sufficiently large.
Closed-loop system is quite robust against model errors!
(High gain control with stable open loop zeros)

Implementation

A relay with hysteresis or a smooth (e.g., linear) region is often used in practice.
Choice of hysteresis or smoothing parameter can be critical for performance
More complicated structures with several relays possible. Harder to design and analyze.

ABS Breaking

By moving along the dashed arrow, an ABS controller attains higher average friction than what is obtained by locked wheels.
Ideally, the slip ratio should be kept at the maximizing value \( \lambda_{\max} \).

Next Lectures

◮ L10–L12: Optimal control methods
◮ L13: Other synthesis methods
◮ L14: Course summary
Next Lecture

- Optimal control

Read chapter 18 in [Glad & Ljung] for preparation.