or

“How to make a circle out of the point $-1 + 0i$, and different ways to stay away from it ...”
| Lecture 1-3 | Modelling and basic phenomena  
|            | (linearization, phase plane, limit cycles) |
| Lecture 4-6 | Analysis methods  
|            | (Lyapunov, circle criterion, describing functions) |
| Lecture 7-8 | Common nonlinearities  
|            | (Saturation, friction, backlash, quantization) |
| Lecture 9-13 | Design methods  
|             | (Lyapunov methods, Backstepping, Optimal control) |
| Lecture 14  | Summary |
Today’s Goal

To understand

- signal norms
- system gain
- bounded input bounded output (BIBO) stability

To be able to analyze stability using

- the Small Gain Theorem,
- the Circle Criterion,
- Passivity

Material

- [Glad & Ljung]: Ch 1.5-1.6, 12.3  [Khalil]: Ch 5–7.1
- lecture slides
For what $G(s)$ and $f(\cdot)$ is the closed-loop system stable?

- Lur’e and Postnikov’s problem (1944)
- Aizerman’s conjecture (1949) (False!)
- Kalman’s conjecture (1957) (False!)
- Solution by Popov (1960) (Led to the Circle Criterion)
Gain

**Idea:** Generalize static gain to nonlinear dynamical systems

The gain \( \gamma \) of \( S \) measures the largest amplification from \( u \) to \( y \)

Here \( S \) can be a constant, a matrix, a linear time-invariant system, a nonlinear system, etc

**Question:** How should we measure the size of \( u \) and \( y \)?
A norm $\| \cdot \|$ measures size.

A **norm** is a function from a space $\Omega$ to $\mathbb{R}^+$, such that for all $x, y \in \Omega$

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $\alpha \in \mathbb{R}$

**Examples**

Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$

Max norm: $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$
A signal \( x(t) \) is a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^d \).
A signal norm is a way to measure the size of \( x(t) \).

**Examples**

2-norm (energy norm):
\[
\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 dt}
\]

sup-norm:
\[
\|x\|_\infty = \sup_{t \in \mathbb{R}^+} |x(t)|
\]

The space of signals with \( \|x\|_2 < \infty \) is denoted \( L_2 \).
Parseval’s Theorem

**Theorem** If \( x, y \in L_2 \) have the Fourier transforms

\[
X(i\omega) = \int_{0}^{\infty} e^{-i\omega t} x(t) dt, \quad Y(i\omega) = \int_{0}^{\infty} e^{-i\omega t} y(t) dt,
\]

then

\[
\int_{0}^{\infty} y^T(t)x(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(i\omega)X(i\omega) d\omega.
\]

In particular

\[
\|x\|_2^2 = \int_{0}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(i\omega)|^2 d\omega.
\]

\( \|x\|_2 < \infty \) corresponds to bounded energy.
A system $S$ is a map between two signal spaces: $y = S(u)$.

$u \xrightarrow{S} y$

The gain of $S$ is defined as

$$\gamma(S) = \sup_{u \in L_2} \frac{\|y\|_2^2}{\|u\|_2^2} = \sup_{u \in L_2} \frac{\|S(u)\|_2^2}{\|u\|_2^2}$$

**Example** The gain of a static relation $y(t) = \alpha u(t)$ is

$$\gamma(\alpha) = \sup_{u \in L_2} \frac{\|\alpha u\|_2^2}{\|u\|_2^2} = \sup_{u \in L_2} \frac{|\alpha| \|u\|_2^2}{\|u\|_2^2} = |\alpha|$$
Example—Gain of a Stable Linear System

\[ \gamma(G) = \sup_{u \in L_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega \in (0, \infty)} |G(i\omega)| \]

**Proof:** Assume \(|G(i\omega)| \leq K\) for \(\omega \in (0, \infty)\). Parseval’s theorem gives

\[
\|\gamma\|^2_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \leq K^2 \|u\|^2_2
\]

This proves that \(\gamma(G) \leq K\). See [Khalil, Appendix C.10] for a proof of the equality.
2 minute exercise: Show that $\gamma(S_1S_2) \leq \gamma(S_1)\gamma(S_2)$. 
Example—Gain of a Static Nonlinearity

\[ |f(x)| \leq K|x|, \quad f(x^*) = Kx^* \]

\[
\|y\|_2^2 = \int_0^\infty f^2(u(t))\,dt \leq \int_0^\infty K^2u^2(t)\,dt = K^2\|u\|_2^2
\]

\( u(t) = x^*, \; t \in (0, \infty) \) gives equality \( \Rightarrow \)

\[ \gamma(f) = \sup_{u \in L^2} \frac{\|y\|_2}{\|u\|_2} = K. \]
BIBO Stability

\[ \gamma(S) = \sup_{u \in L^2} \frac{\|y\|^2}{\|u\|^2} \]

**Definition**

\( S \) is bounded-input bounded-output (BIBO) stable if \( \gamma(S) < \infty \).

**Example:** If \( \dot{x} = Ax \) is asymptotically stable then \( G(s) = C(sI - A)^{-1}B + D \) is BIBO stable.
**Theorem**

Assume $S_1$ and $S_2$ are BIBO stable. If

$$\gamma(S_1)\gamma(S_2) < 1$$

then the closed-loop map from $(r_1, r_2)$ to $(e_1, e_2)$ is BIBO stable.
"Proof" of the Small Gain Theorem

Existence of solution \((e_1, e_2)\) for every \((r_1, r_2)\) has to be verified separately. Then

\[
\|e_1\|_2 \leq \|r_1\|_2 + \gamma(S_2)[\|r_2\|_2 + \gamma(S_1)\|e_1\|_2]
\]

gives

\[
\|e_1\|_2 \leq \frac{\|r_1\|_2 + \gamma(S_2)\|r_2\|_2}{1 - \gamma(S_2)\gamma(S_1)}
\]

\(\gamma(S_2)\gamma(S_1) < 1, \|r_1\|_2 < \infty, \|r_2\|_2 < \infty\) give \(\|e_1\|_2 < \infty\). Similarly we get

\[
\|e_2\|_2 \leq \frac{\|r_2\|_2 + \gamma(S_1)\|r_1\|_2}{1 - \gamma(S_1)\gamma(S_2)}
\]

so also \(e_2\) is bounded.
Linear System with Static Nonlinear Feedback (1)

\[
G(s) = \frac{2}{(s + 1)^2} \quad \text{and} \quad 0 \leq \frac{f(y)}{y} \leq K
\]

\[
\gamma(G) = 2 \quad \text{and} \quad \gamma(f) \leq K.
\]

The small gain theorem gives that \( K \in [0, 1/2) \) implies BIBO stability.
The Nyquist Theorem

Theorem
The closed loop system is stable iff the number of counter-clockwise encirclements of $-1$ by $G(\Omega)$ (note: $\omega$ increasing) equals the number of open loop unstable poles.
Let $f(y) = K y$ for the previous system.

The Nyquist Theorem proves stability when $K \in [0, \infty)$. The Small Gain Theorem proves stability when $K \in [0, 1/2)$. 
The Circle Criterion

Case 1: $0 < k_1 \leq k_2 < \infty$

\[
\begin{align*}
\text{Theorem} & \quad \text{Consider a feedback loop with } y = Gu \text{ and } \\
& \quad u = -f(y) + r. \text{ Assume } G(s) \text{ is stable and that} \\
& \quad 0 < k_1 \leq \frac{f(y)}{y} \leq k_2. \\
& \quad \text{If the Nyquist curve of } G(s) \text{ does not intersect or encircle the} \\
& \quad \text{circle defined by the points } -1/k_1 \text{ and } -1/k_2, \text{ then the} \\
& \quad \text{closed-loop system is BIBO stable from } r \text{ to } y.
\end{align*}
\]
Other cases

\( G: \) stable system

- 0 < \( k_1 < k_2 \): Stay outside circle
- 0 = \( k_1 < k_2 \): Stay to the right of the line \( \text{Re } s = -1/k_2 \)
- \( k_1 < 0 < k_2 \): Stay inside the circle

Other cases: Multiply \( f \) and \( G \) with \(-1\).

\( G: \) Unstable system

To be able to guarantee stability, \( k_1 \) and \( k_2 \) must have same sign (otherwise unstable for \( k = 0 \))

- 0 < \( k_1 < k_2 \): Encircle the circle \( p \) times counter-clockwise (if \( \omega \) increasing)
- \( k_1 < k_2 < 0 \): Encircle the circle \( p \) times counter-clockwise (if \( \omega \) increasing)
The “circle” is defined by $-1/k_1 = -\infty$ and $-1/k_2 = -1/K$.

$$\min \text{Re } G(i\omega) = -1/4$$

so the Circle Criterion gives that if $K \in [0, 4)$ the system is BIBO stable.
Proof of the Circle Criterion

Let $k = (k_1 + k_2)/2$ and $\tilde{f}(y) = f(y) - ky$. Then

$$\left| \frac{\tilde{f}(y)}{y} \right| \leq \frac{k_2 - k_1}{2} =: R$$

$$\tilde{r}_1 = r_1 - kr_2$$
Proof of the Circle Criterion (cont’d)

\[ \tilde{r}_1 \rightarrow \tilde{G}(s) \rightarrow r_2 \]

SGT gives stability for \(|\tilde{G}(i\omega)|R < 1\) with \(\tilde{G} = \frac{G}{1 + kG}\).

\[ R < \frac{1}{|\tilde{G}(i\omega)|} = \left| \frac{1}{G(i\omega)} + k \right| \]

Transform this expression through \(z \rightarrow 1/z\).
Lyapunov revisited

Original idea: “Energy is decreasing”

\[
\dot{x} = f(x), \quad x(0) = x_0 \\
V(x(T)) - V(x(0)) \leq 0 \\
(\text{+some other conditions on } V)
\]

New idea: “Increase in stored energy \leq \text{added energy}”

\[
\dot{x} = f(x, u), \quad x(0) = x_0 \\
y = h(x) \\
V(x(T)) - V(x(0)) \leq \int_0^T \varphi(y, u) \, dt \tag{1}
\]

external power
Motivation

Will assume the external power has the form \( \phi(y, u) = y^T u \).

Only interested in BIBO behavior. Note that

\[
\exists V \geq 0 \text{ with } V(x(0)) = 0 \text{ and } (1) \iff \\
\int_0^T y^T u \, dt \geq 0
\]

Motivated by this we make the following definition
Definition The system $S$ is passive from $u$ to $y$ if

$$\int_{0}^{T} y^T u \, dt \geq 0,$$

for all $u$ and all $T > 0$

and strictly passive from $u$ to $y$ if there $\exists \epsilon > 0$ such that

$$\int_{0}^{T} y^T u \, dt \geq \epsilon (|y|_T^2 + |u|_T^2),$$

for all $u$ and all $T > 0$
Define the **scalar product**

\[ \langle y, u \rangle_T = \int_0^T y^T(t)u(t) \, dt \]

**Cauchy-Schwarz** inequality:

\[ \langle y, u \rangle_T \leq |y|_T |u|_T \]

where \(|y|_T = \sqrt{\langle y, y \rangle_T}\). Note that \(|y|_\infty = \|y\|_2\).
2 minute exercise:
Feedback of Passive Systems is Passive

If $S_1$ and $S_2$ are passive, then the closed-loop system from $(r_1, r_2)$ to $(y_1, y_2)$ is also passive.

Proof: 

$$\langle y, r \rangle_T = \langle y_1, r_1 \rangle_T + \langle y_2, r_2 \rangle_T$$

$$= \langle y_1, r_1 - y_2 \rangle_T + \langle y_2, r_2 + y_1 \rangle_T$$

$$= \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T \geq 0$$

Hence, $\langle y, r \rangle_T \geq 0$ if $\langle y_1, e_1 \rangle_T \geq 0$ and $\langle y_2, e_2 \rangle_T \geq 0$
Theorem  An asymptotically stable linear system $G(s)$ is passive if and only if

$$\text{Re} \ G(i\omega) \geq 0, \quad \forall \omega > 0$$

It is strictly passive if and only if there exists $\epsilon > 0$ such that

$$\text{Re} \ G(i\omega) \geq \epsilon(1 + |G(i\omega)|^2), \quad \forall \omega > 0$$

Example

$G(s) = \frac{s + 1}{s + 2}$ is passive and strictly passive,

$G(s) = \frac{1}{s}$ is passive but not strictly passive.
A Strictly Passive System Has Finite Gain

If $S$ is strictly passive, then $\gamma(S) < \infty$.

Proof: Note that $\|y\|_2 = \lim_{T \to \infty} |y|_T$.

$$\epsilon(|y|_T^2 + |u|_T^2) \leq \langle y, u \rangle_T \leq |y|_T \cdot |u|_T \leq \|y\|_2 \cdot \|u\|_2$$

Hence, $\epsilon |y|_T^2 \leq \|y\|_2 \cdot \|u\|_2$, so letting $T \to \infty$ gives

$$\|y\|_2 \leq \frac{1}{\epsilon} \|u\|_2$$
The Passivity Theorem

**Theorem** If $S_1$ is strictly passive and $S_2$ is passive, then the closed-loop system is BIBO stable from $r$ to $y$. 
Proof of the Passivity Theorem

$S_1$ strictly passive and $S_2$ passive give

$$
\epsilon\left(|y_1|^2_T + |e_1|^2_T\right) \leq \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T = \langle y, r \rangle_T
$$

Therefore

$$
|y_1|^2_T + \langle r_1 - y_2, r_1 - y_2 \rangle_T \leq \frac{1}{\epsilon} \langle y, r \rangle_T
$$

or

$$
|y_1|^2_T + |y_2|^2_T - 2\langle y_2, r_2 \rangle_T + |r_1|^2_T \leq \frac{1}{\epsilon} \langle y, r \rangle_T
$$

Finally

$$
|y|^2_T \leq 2\langle y_2, r_2 \rangle_T + \frac{1}{\epsilon} \langle y, r \rangle_T \leq \left(2 + \frac{1}{\epsilon}\right)|y|^T|r|^T
$$

Letting $T \to \infty$ gives $\|y\|_2 \leq C\|r\|_2$ and the result follows.
Passivity Theorem is a “Small Phase Theorem”
Example—Gain Adaptation

Applications in channel estimation in telecommunication, noise cancelling etc.

\[
\frac{d\theta}{dt} = -\gamma u(t)[y_m(t) - y(t)], \quad \gamma > 0.
\]
Gain Adaptation—Closed-Loop System

\[ u - \gamma s \theta = (G(s) \theta^*)(t) - G(s) y_m \]

\[ \frac{\gamma}{s} \]
Gain Adaptation is BIBO Stable

\[ (\theta - \theta^*) u \rightarrow G(s) \rightarrow y_m - y \]

\[ S \text{ is passive (Exercise 4.12), so the closed-loop system is BIBO stable if } G(s) \text{ is strictly passive.} \]
Simulation of Gain Adaptation

Let $G(s) = \frac{1}{s+1} + \epsilon$, $\gamma = 1$, $u = \sin t$, $\theta(0) = 0$ and $\gamma^* = 1$
Storage Function

Consider the nonlinear control system

\[ \dot{x} = f(x, u), \quad y = h(x) \]

A **storage function** is a \( C^1 \) function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) such that

- \( V(0) = 0 \) and \( V(x) \geq 0, \quad \forall x \neq 0 \)
- \( \dot{V}(x) \leq u^T y, \quad \forall x, u \)

**Remark:**

- \( V(T) \) represents the stored energy in the system

\[ \sqrt{V(x(T))} \leq \int_0^T y(t)u(t)dt + V(x(0)), \quad \forall T > 0 \]
Lemma: If there exists a storage function $V$ for a system

$$\dot{x} = f(x, u), \quad y = h(x)$$

with $x(0) = 0$, then the system is passive.

Proof: For all $T > 0$,

$$\langle y, u \rangle_T = \int_0^T y(t)u(t)dt \geq V(x(T)) - V(x(0)) = V(x(T)) \geq 0$$
Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

Lyapunov idea: “Energy is decreasing”

\[ \dot{V} \leq 0 \]

Passivity idea: “Increase in stored energy \leq \text{Added energy}”

\[ \dot{V} \leq u^T y \]
Example KYP Lemma

Consider an asymptotically stable linear system

\[ \dot{x} = Ax + Bu, \ y = Cx \]

Assume there exists positive definite symmetric matrices \( P, Q \) such that

\[ A^T P + PA = -Q, \ \text{and} \ B^T P = C \]

Consider \( V = 0.5x^T P x \). Then

\[ \dot{V} = 0.5(\dot{x}^T P x + x^T P \dot{x}) = 0.5x^T (A^T P + PA)x + u^T B^T P x \]
\[ = -0.5x^T Q x + u^T y < u^T y, \ x \neq 0 \]  \hspace{1cm} (2)

and hence the system is strictly passive. This fact is part of the Kalman-Yakubovich-Popov lemma.
Describing functions (analysis of oscillations)