Lecture 7: Input-Output Models

[IFAC PB pg 23-35]

- Shift operators; the pulse transfer operator
- Z-transform; the pulse transfer function
- System response
- Poles and zeros
- Transformations between system representations

Linear System Models

- State-space model
  \[ x(t) = A x(t) + B u(t) \]
  \[ y(t) = C x(t) \]
- Input-output models
  \[ G(p) / G(s) \]

More I-O models: pulse response, step response, frequency function, . . .

Shift Operators

Operators on time series

Assume \( h = 1 \) (the sampling-time convention)

Time series are doubly infinite sequences:

- \( f(k) : k = \ldots -1,0,1,\ldots \)

Forward shift operator:

- denoted \( q \)
- \( q f(k) = f(k + 1) \)
- \( q^n f(k) = f(k + n) \)

Pulse Transfer Operator

Rewrite the state-space model using the forward shift operator:

\[
\begin{align*}
x(k+1) &= q x(k) = \Phi x(k) + \Gamma u(k) \\
y(k) &= C x(k) + D u(k)
\end{align*}
\]

Eliminate \( x(k) \):

\[
\begin{align*}
x(k) &= (qI - \Phi)^{-1} \Gamma u(k) \\
y(k) &= C x(k) + D u(k) = C (qI - \Phi)^{-1} \Gamma u(k) + D u(k) \\
&= [C(qI - \Phi)^{-1} \Gamma + D] u(k) = H(q) u(k)
\end{align*}
\]

\( H(q) \) is the pulse transfer operator of the system

Describes how the input and output are related.

Shift Operators

Backward shift operator:

- denoted \( q^{-1} \)
- \( q^{-1} f(k) = f(k - 1) \)
- \( q^n f(k) = f(k - n) \)

Poles and Zeros (SISO case)

The pulse transfer function can be written as a rational function

\[ H(q) = \frac{B(q)}{A(q)} \]

deg \( A = n \) = the number of states

deg \( B \leq n \) = \( n_s \)

\( A(q) \) is the characteristic polynomial of \( \Phi \), i.e.

\[ A(q) = \det(qI - \Phi) \]

The poles of the system are given by \( A(q) = 0 \)
The zeros of the system are given by \( B(q) = 0 \)
Disk Drive Example

Recall the double integrator from the previous lecture:

\[
\begin{align*}
\frac{dx}{dt} & = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y & = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{align*}
\]

Sample with \( h = 1 \):

\[
\begin{align*}
\Phi & = e^{Ah} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
\Gamma & = \int_0^h e^{A(t-h)} B \, dt = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}
\end{align*}
\]

From Pulse Transfer Operator to Difference Equation

\[
y(k) = H(q)u(k)
\]

\[
A(q)y(k) = B(q)u(k)
\]

\[
(q^n + a_1q^{n-1} + \cdots + a_n) y(k) = (b_0q^n + \cdots + b_n) u(k)
\]

which means

\[
y(k + n) + a_1y(k + n - 1) + \cdots + a_n y(k) = b_0u(k + n) + \cdots + b_n u(k)
\]

The reciprocal polynomial

\[
A^*(q) = 1 + a_1q + \cdots + a_nq^n = q^nA(q^{-1})
\]

is obtained from the polynomial \( A \) by reversing the order of the coefficients.

Now the system can instead be written as

\[
A^*(q^{-1})y(k) = B^*(q^{-1})u(k - d)
\]

Disk Drive Example cont.

Pulse transfer operator:

\[
H(q) = C(qI - \Phi)^{-1}\Gamma + D
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} q-1 & 1 \\ 0 & q-1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
= \frac{0.5(q+1)}{(q-1)^2}
\]

Two poles in 1, one zero in \(-1\).

Difference Equation with Backward Shift

\[
y(k + n) + a_1y(k + n - 1) + \cdots + a_n y(k)
\]

\[
= b_0u(k + n) + \cdots + b_n u(k)
\]

can be written as

\[
y(k) + a_1y(k - 1) + \cdots + a_n y(k - n)
\]

\[
= b_0u(k - d) + \cdots + b_n u(k - d - n)
\]

where \( d = n - n_0 \) is the pole excess of the system.

Difference Equation Example

Using forward shift

\[
y(k + 2) + 2y(k + 1) + 3y(k) = 2u(k + 1) + u(k)
\]

\[
\begin{align*}
(q^2 + 2q + 3)y(k) & = (2q + 1)u(k) \\
A(q) & = q^2 + 2q + 3 \\
B(q) & = 2q + 1
\end{align*}
\]

Hence,

\[
A(q) = q^2 + 2q + 3 \\
B(q) = 2q + 1
\]
Difference Equation Example, continued

Using backward shift
\[ y(k) + 2y(k - 1) + 3y(k - 2) = 2u(k - 1) + u(k - 2) \]
can be written \((d = 1)\)
\[ (1 + 2q^{-1} + 3q^{-2})y(k) = (2 + q^{-1})u(k - 1) \]
Hence,
\[ A(q^{-1}) = 1 + 2q^{-1} + 3q^{-2} \]
\[ B(q^{-1}) = 2 + q^{-1} \]

Example — Discrete-Time Step Signal

Let \( y(k) = 1 \) for \( k \geq 0 \). Then
\[ Y(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z-1}, \quad |z| > 1 \]

Application of the following result for power series
\[ \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1 \]

Example — Discrete-Time Ramp Signal

Let \( y(k) = k \) for \( k \geq 0 \). Then
\[ Y(z) = 0 + z^{-1} + 2z^{-2} + 3z^{-3} + \cdots = \frac{z}{(z-1)^2} \]

Application of the following result for power series
\[ \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \text{ for } |x| < 1 \]

Z-transform Table

Table 2 (pg 26) in IFAC PB (ignore the middle column!)

<table>
<thead>
<tr>
<th>( f )</th>
<th>( \mathcal{Z}f )</th>
<th>( \mathcal{Z}f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(k) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( 1 \ k \geq 0 )</td>
<td>( \frac{1}{z} )</td>
<td>( \frac{1}{z^2} )</td>
</tr>
<tr>
<td>( k )</td>
<td>( \frac{k}{z} )</td>
<td>( \frac{k^2z}{z-1} )</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2n} )</td>
<td>( \frac{1}{2(z-1)^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} )</td>
<td>( \frac{1}{1-z^{-1}} )</td>
<td>( \frac{1}{z-z^{-1}} )</td>
</tr>
<tr>
<td>( 1 - e^{-\alpha t} )</td>
<td>( \frac{1}{z(z-1)} )</td>
<td>( \frac{z(z-1)}{z-1} )</td>
</tr>
<tr>
<td>( \sin \alpha k )</td>
<td>( \frac{\alpha}{z^2 + \omega^2} )</td>
<td>( \frac{\sin \alpha}{z^2 + 2\zeta \omega z + \omega^2} )</td>
</tr>
</tbody>
</table>

Some Properties of the Z-transform

\[ \mathcal{Z}(\alpha f + \beta g) = \alpha F(z) + \beta G(z) \]
\[ \mathcal{Z}(q^{-n} f) = z^{-n} F(z) \]
\[ \mathcal{Z}(q f) = z(F(z) - f(0)) \]
\[ \mathcal{Z}(f * g) = \mathcal{Z} \left( \sum_{j=0}^{\infty} f(j)g(k-j) \right) = F(z)G(z) \]
From State Space to Pulse Transfer Function

\[
\begin{align*}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= Cx(k) + Du(k)
\end{align*}
\]

\[
\begin{align*}
x(X(z) - x(0)) &= \Phi X(z) + \Gamma U(z) \\
y(z) &= CX(z) + DU(z)
\end{align*}
\]

\[
Y(z) = C(zI - \Phi)^{-1}x(0) + [C(zI - \Phi)^{-1}\Gamma + D]U(z)
\]

The rational function \(H(z) = C(zI - \Phi)^{-1}\Gamma + D\) is called the \textit{pulse transfer function} from \(u\) to \(y\).

It is the Z-transform of the pulse response.

\[H(q) \text{ vs } H(z)\]

The pulse transfer operator \(H(q)\) and the pulse transfer function \(H(z)\) are the same rational functions. They have the same poles and zeros.

\(H(q)\) is used in the time domain (\(q = \text{shift operator}\))

\(H(z)\) is used in the Z-domain (\(z = \text{complex variable}\))

Calculating System Response Using the Z-transform

1. Find the pulse transfer function \(H(z) = C(zI - \Phi)^{-1}\Gamma + D\)
2. Compute the Z-transform of the input: \(U(z) = Z\{u(k)\}\)
3. Compute the Z-transform of the output:
   \[
   Y(z) = C(zI - \Phi)^{-1}x(0) + H(z)U(z)
   \]
4. Apply the inverse Z-transform (table) to find the output:
   \[
   y(k) = Z^{-1}\{Y(z)\}
   \]

Calculating Pulse Response Using the Z-transform

1. Find the pulse transfer function \(H(z) = C(zI - \Phi)^{-1}\Gamma + D\)
2. The Z-transform of a pulse: \(U(z) = 1\)
3. Compute the Z-transform of the output (assuming \(x(0) = 0\)):
   \[
   Y(z) = H(z)
   \]
4. Apply the inverse Z-transform (table) to find the output:
   \[
   h(k) = Z^{-1}\{Y(z)\}
   \]

Calculating Pulse Response directly from \(H(z)\)

Example:
Assume that \(H(z) = \frac{1}{z^2}\)
This can be directly written as the difference equation
\[
y(k+1) = 0.5y(k) + u(k+1)
\]
from which it directly follows that
\[
\begin{align*}
h(0) &= u(0) = 1 \\
h(1) &= 0.5h(0) + u(1) = 0.5 \\
h(2) &= 0.5^2 \\
h(k) &= 0.5^k
\end{align*}
\]

Frequency Response in Continuous Time

Given a stable system \(G(s)\), the input \(u(t) = \sin \omega t\) will, after a transient, give the output
\[
y(t) = |G(i\omega)| \sin \left(\omega t + \arg G(i\omega)\right)
\]
- The amplitude and phase shift for different frequencies are given by the value of \(G(s)\) along the imaginary axes, i.e. \(G(i\omega)\)
- Plotted in Bode and Nyquist diagrams.
Given a stable system $H(z)$, the input $u(k) = \sin(\omega k)$ will, after a transient, give the output

$$y(k) = |H(e^{\jmath \omega})| \sin (\omega k + \text{arg } H(e^{\jmath \omega}))$$

- $G(s)$ and the imaginary axis are replaced by $H(z)$ and the unit circle.
- Only describes what happens at the sampling instants
- The inter-sample behavior is not studied in this course

For slow signals, the hold circuit is approximately a $\frac{h}{2}$ delay.
For fast signals, the hold circuit destroys the sinusoidal shape.

- **Nyquist diagram** for cont. transfer function $1/(s^2 + 1.4s + 1)$ (solid) and for ZOH-sampled counterpart (dashed, plotted for $\omega h \in [0, \pi]$)

- **Interpretation of Poles and Zeros**
  - Poles:
    - A pole $z = a$ is associated with the time function $f(k) = a^k$
  - Zeros:
    - A zero $z = a$ implies that the transmission of the input $u(k) = a^k$ is blocked by the system
    - Related to how inputs and outputs are coupled to the states

- **Transformation of Poles via Sampling**: $z_i = e^{\jmath \omega_i h}$

- **New Evidence of the Alias Problem**
  - Several points in the $s$-plane are mapped into the same point in the $z$-plane. The map is not bijective
Sampling of a Second Order System

\[ G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad |\zeta| < 1 \]

- Larger \( \omega_0 \Rightarrow \) faster system response
- Smaller \( \varphi \Rightarrow \) larger damping (relative damping \( \zeta = \cos \varphi \)).
  (Common design choice: \( \zeta = \cos 45^\circ \approx 0.7 \))

Transformation of Zeros via Sampling

- More complicated than for poles
- Extra zeros may appear in the sampled system
- There can be zeros outside the unit circle (non-minimum phase) even if the continuous system has all the zeros in the left half plane
- For short sampling periods
  \[ z_i \approx e^{\lambda_h} \]

Calculation of \( H(z) \) Given \( G(s) \)

Three approaches:
1. Make state-space realization of \( G(s) \). Sample to get \( \Phi \) and \( \Gamma \). Then \( H(z) = C(zI - \Phi)^{-1}\Gamma + D \).
2. Directly using the formula
   \[
   H_{\text{ZOH}}(z) = \sum_{i=0} \frac{1}{z - z_i} \mathcal{L}^{-1} \left[ \frac{1}{s} \right] G(s) \text{ d}s
   \]
   where
   \[
   a_1 = -2e^{-\zeta\omega_0 h} \cos \left( \sqrt{1 - \zeta^2} \omega_0 h \right) \\
   a_2 = e^{-2\zeta\omega_0 h}
   \]

- \( z_i \) are the poles of \( G(s) \) and \( \text{Res} \) denotes the residue.
- outside the scope of the course
3. Use Table 3 (pg 28) in IFAC PB

<table>
<thead>
<tr>
<th>$G(s)$</th>
<th>$H(z)$ = $\frac{b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n}{s^m + a_1 s^{m-1} + \ldots + a_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{z}$</td>
<td>$H(z) = \frac{1}{z-1}$</td>
</tr>
<tr>
<td>$\frac{1}{z^2}$</td>
<td>$H(z) = \frac{h}{2(z-1)^2}$</td>
</tr>
<tr>
<td>$e^{-ah}$</td>
<td>$H(z) = \frac{1 - \exp(-ah)}{z - \exp(-ah)}$</td>
</tr>
<tr>
<td>$\frac{a}{s + a}$</td>
<td>$H(z) = \frac{b_1}{s + a} (1 + e^{-ah})$</td>
</tr>
<tr>
<td>$\frac{a}{(s + a)^2}$</td>
<td>$H(z) = \frac{b_1}{s + a} (1 + e^{-ah})$</td>
</tr>
</tbody>
</table>

Examples in Matlab

```matlab
>> Phi = [0.5 -0.2; 0 0];
> Gamma = [2; 1];
> C = [1 0];
> D = 0;
> h = 1;
> H = ss(Phi, Gamma, C, D, h);
>> zpk(H)
>> % From cont-time transfer function to discrete-time
>> % pulse transfer function
>> Q = 1/s^3;
>> H = c2d(Q, h);
>> % Another way
>> G = tf([1],[1 3 2 0]);
> Q = ss(G);
> H = c2d(G, h);
>> tf(H)
```

Calculation of $H(z)$ Given $G(s)$

Example: For $G(s) = e^{-\tau s}/s^2$, the previous lecture gave

$x(kh + h) = \Phi x(kh) + \Gamma_1 u(kh - h) + \Gamma_0 u(kh)$

$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} \tau \left( \frac{h}{2} - \frac{1}{2} \right) \\ \tau \left( \frac{h}{2} - \frac{1}{2} \right) \end{pmatrix}$, $\Gamma_0 = \begin{pmatrix} (h - \tau)^2 \\ 2(h - \tau) \end{pmatrix}$

With $h = 1$ and $\tau = 0.5$, this gives

$H(z) = C(\Phi - I)^{-1}(\Gamma_0 + \Gamma_1 z^{-1}) = \frac{0.125(z^2 + 6z + 1)}{(z^2 - 2z + 1)}$

Order: 3
Poles: 0, 1, and 1
Zeros: $-3 \pm \sqrt{8}$