Lecture 7: Input-Output Models

[IFAC PB pg 24-33]

- Shift operators
- Pulse transfer operator
- Z-transform
- Pulse transfer function
- Poles and zeros
- Transformations between system representations

Shift Operators

Operators on time series
Assume \( h = 1 \) (the sampling-time convention)
Time series are doubly infinite sequences:
- \( f(k) : k = \ldots -1,0,1,\ldots \)

Forward shift operator:
- denoted \( q \)
- \( qf(k) = f(k+1) \)
- \( q^n f(k) = f(k+n) \)

Backward shift operator:
- denoted \( q^{-1} \)
- \( q^{-1} f(k) = f(k-1) \)
- \( q^{-n} f(k) = f(k-n) \)

Pulse Transfer Operator

Rewrite the state-space model using the forward shift operator:
\[
x(k+1) = q\Phi x(k) + \Gamma u(k) \\
y(k) = Cx(k) + Du(k)
\]
Eliminate \( x(k) \), (assuming \( x(0) = 0 \)):
\[
x(k) = (qI - \Phi)^{-1} \Gamma u(k) \\
y(k) = C(qI - \Phi)^{-1} \Gamma u(k) + D u(k) = H(q) u(k)
\]

\( H(q) \) is the pulse transfer operator of the system
Describes how the input and output are related.

Poles and Zeros (SISO case)
The pulse transfer function can be written as a rational function
\[
H(q) = \frac{B(q)}{A(q)}
\]
- \( \deg A = n \) = the number of states
- \( \deg B = n_z \leq n \)
- \( A(q) \) is the characteristic polynomial of \( \Phi \), i.e.
\[
A(q) = \det(qI - \Phi)
\]
The poles of the system are given by \( A(q) = 0 \)
The zeros of the system are given by \( B(q) = 0 \)

Disk Drive Example

Recall the double integrator from the previous lecture:
\[
\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]
Sample with \( h = 1 \):
\[
\Phi = e^{Ah} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
\Gamma = \int_0^h e^{As} B \, ds = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}
\]
Hence,

Using forward shift

\[ H(q) = C(qI - \Phi)^{-1} \Gamma + D \]

\[
= [1 \ 0 \ 0 \ q - 1 -1]^{-1} [0.5] = [1 \ 0 \ 0 \ q - 1 -1] \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}
\]

\[= 0.5(q + 1) \]

Two poles in 1 and one zero in -1.

\[\]

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Disk Drive Example cont.

Pulse transfer operator:

\[ H(q) = C(qI - \Phi)^{-1} \Gamma + D \]

\[= [1 \ 0 \ 0 \ q - 1 -1]^{-1} [0.5] = [1 \ 0 \ 0 \ q - 1 -1] \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \]

\[= 0.5(q + 1) \]

Two poles in 1 and one zero in -1.

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From Pulse Transfer Operator to Difference Equation

\[ y(k) = H(q)u(k) \]

\[ A(q)y(k) = B(q)u(k) \]

\[ (q^n + a_{n-1}q^{n-1} + \cdots + a_0)y(k) = (b_0q^n + \cdots + b_n)u(k) \]

which means

\[ y(k + n) + a_1y(k + n - 1) + \cdots + a_ny(k) = b_0u(k + n) + \cdots + b_nu(k) \]

This form is also known as the input-output form

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Input-Output Form with Backward Shift

\[ y(k + n) + a_1y(k + n - 1) + \cdots + a_ny(k) \]

\[= b_0u(k + n) + \cdots + b_nu(k) \]

can be written as

\[ y(k) + a_1y(k - 1) + \cdots + a_ny(k - n) \]

\[= b_0u(k - d) + \cdots + b_nu(k - d - n) \]

where \(d = n - n_b\) is the pole excess of the system (the number of pure time delays in the system)

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Input-Output Example

Using forward shift

\[ y(k + 2) + 2y(k + 1) + 3y(k) = 2u(k + 1) + u(k) \]

can be written

\[ (q^2 + 2q + 3)y(k) = (2q + 1)u(k) \]

Hence,

\[ A(q) = q^2 + 2q + 3 \]

\[ B(q) = 2q + 1 \]

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Input-Output Example, continued

Using backward shift

\[ y(k) + 2y(k - 1) + 3y(k - 2) = 2u(k - 1) + u(k - 2) \]

can be written \(d = 1\)

\[ (1 + 2q^{-1} + 3q^{-2})y(k) = (2 + q^{-1})u(k - 1) \]

Hence,

\[ A(q^{-1}) = 1 + 2q^{-1} + 3q^{-2} \]

\[ B(q^{-1}) = 2 + q^{-1} \]
**Z-transform**

The discrete-time counterpart to the Laplace transform

Defined on semi-infinite time series \( f(k) : k = 0, 1, \ldots \)

\[
Z(f(k)) = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}
\]

\( z \) is a complex variable

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**Example — Discrete-Time Step Signal**

Let \( y(k) = 1 \) for \( k \geq 0 \). Then

\[
Y(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z-1}, \quad |z| > 1
\]

Application of the following result for power series

\[
\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for} \quad |x| < 1
\]

---

**Example — Discrete-Time Ramp Signal**

Let \( y(k) = k \) for \( k \geq 0 \). Then

\[
Y(z) = z^{-1} + z^{-2} + 2z^{-3} + \cdots = \frac{z}{(z-1)^2}
\]

Application of the following result for power series

\[
\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{for} \quad |x| < 1
\]

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**Z-transform Table**

<table>
<thead>
<tr>
<th>( f(k) )</th>
<th>( Z(f(k)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(k) ) (pulse)</td>
<td>1</td>
</tr>
<tr>
<td>1 (step)</td>
<td>( \frac{z}{z-1} )</td>
</tr>
<tr>
<td>( k ) (ramp)</td>
<td>( \frac{z}{(z-1)^2} )</td>
</tr>
<tr>
<td>( a^k )</td>
<td>( \frac{z}{z-a} )</td>
</tr>
</tbody>
</table>

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**Some Properties of the Z-transform**

\( Z \) linear

\[
Z(q^n f) = z^n F(z)
\]

\[
Z(q f) = z(F(z) - f(0))
\]

\[
Z(f * g) = \sum_{j=0}^{k} f(j)g(k-j) = (Zf) \cdot (Zg)
\]

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**From State Space to Pulse Transfer Function**

\[
\begin{aligned}
\{ z(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= C x(k) + D u(k) \\
\end{aligned}
\]

\[
\begin{aligned}
\{ z(X(z) - x(0)) &= \Phi X(z) + \Gamma U(z) \\
Y(z) &= C X(z) + D U(z) \\
\end{aligned}
\]

\[
Y(z) = C(zI - \Phi)^{-1}z x(0) + [C(zI - \Phi)^{-1} \Gamma + D] U(z)
\]

The rational function \( H(z) = C(zI - \Phi)^{-1} \Gamma + D \) is called the pulse transfer function from \( u \) to \( y \).

It is the Z-transform of the pulse response.
The pulse transfer operator $H(q)$ and the pulse transfer function $H(z)$ are the same rational functions. They have the same poles and zeros. $H(q)$ is used in the time domain ($q$ = shift operator) and $H(z)$ is used in the Z-domain ($z$ = complex variable).

Calculating System Response Using the Z-transform

1. Find the pulse transfer function $H(z) = C(zI - \Phi)^{-1} \Gamma + D$
2. Compute the Z-transform of the input: $U(z) = Z\{u(k)\}$
3. Compute the Z-transform of the output:
   $$Y(z) = C(zI - \Phi)^{-1} z(0) + H(z)U(z)$$
4. Apply the inverse Z-transform (table) to find the output:
   $$y(k) = Z^{-1}\{Y(z)\}$$

Frequency Response in Continuous Time

Given a stable system $G(s)$, the input $u(t) = \sin \omega t$ will, after a transient, give the output:
$$y(t) = |G(i\omega)| \sin \left( \omega t + \arg G(i\omega) \right)$$
- The amplitude and phase shift for different frequencies are given by the value of $G(s)$ along the imaginary axes, i.e. $G(i\omega)$.
- Plotted in Bode and Nyquist diagrams.

Bode diagram for continuous transfer function $1/(s^2 + 1.4s + 1)$ (solid) and for ZOH-sampled counterpart (dashed, plotted for $\omega h \in [0, \pi]$)

Nyquist diagram for cont. transfer function $1/(s^2 + 1.4s + 1)$ (solid) and for ZOH-sampled counterpart (dashed, plotted for $\omega h \in [0, \pi]$)

For slow signals, the hold circuit is approximately a $h/2$ delay. For fast signals, the hold circuit destroys the sinusoidal shape.
**Interpretation of Poles and Zeros**

**Poles:**
- A pole $z = a$ is associated with the time function $f(k) = a^k$

**Zeros:**
- A zero $z = a$ implies that the transmission of the input $u(k) = a^k$ is blocked by the system
- Related to how inputs and outputs are coupled to the states

**Transformation of Poles:** $z_i = e^{s_i h}$

**New Evidence of the Alias Problem**

Several points in the $s$-plane are mapped into the same point in the $z$-plane. The map is not bijective

**Sampling of a Second Order System**

The poles of the sampled system are given by $z^2 + a_1 z + a_2 = 0$

where

$$
  a_1 = -2 e^{-\zeta \omega_0 h} \cos \left( \sqrt{1 - \zeta^2} \omega_0 h \right) \\
  a_2 = e^{-2 \zeta \omega_0 h}
$$

**Transformation of Zeros**

- More complicated than for poles
- Extra zeros may appear in the sampled system
- There can be zeros outside the unit circle also if the continuous system has all the zeros in the left half plane
- For short sampling periods $z_i \approx e^{s_i h}$
Calculation of \( H(z) \) Given \( G(s) \)

Three approaches:

1. Make state realization of \( G(s) \). Sample to get \( \Phi \) and \( \Gamma \). Then \( H(z) = C(zI - \Phi)^{-1}\Gamma + D \).

2. Directly using the formula

\[
H(z) = \frac{1}{z} \int_{-\infty}^{+\infty} e^{zt} G(s) \, ds \cdot \frac{1}{z - \alpha}
\]

\[
= \sum_{\alpha_k} \frac{1}{z - \alpha_k} \text{Res} \left\{ e^{zt} - 1 \right\}
\]

- \( \alpha_k \) are the poles of \( G(s) \) and \( \text{Res} \) denotes the residue.
- outside the scope of the course

3. Use Table 2 (pg 28) in IFAC PB

- With care: The table gives the pulse transfer function for the ZOH-sampled transfer function \( G(s) \). It does not give you the Z-transform for the signal with the Laplace transform \( G(s) \).

\[
\frac{G(s)}{s} \begin{cases} 
\frac{h^2(z+1)}{2(z-1)^2} 
\end{cases}
\]

For example, the Laplace transform of a ramp is \( 1/s^2 \). However, the Z-transform of a ramp is not

\[
\frac{h^2(z+1)}{2(z-1)^2}
\]

(Do you remember what it is?)

Calculation of \( H(z) \) Given \( G(s) \)

Example 1. For \( G(s) = 1/s^2 \), the previous lecture gave

\[
\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} h^2 \\ h \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}
\]

With \( h = 1 \), this gives

\[
H(z) = C(zI - \Phi)^{-1}\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & -h \\ 0 & z \end{pmatrix}^{-1} \begin{pmatrix} h^2 \\ h \end{pmatrix} = \frac{h^2(z+1)}{2(z-1)^2}
\]

Example 2. For \( G(s) = e^{-ts}/s^2 \), the previous lecture gave

\[
x(kh + h) = \Phi x(kh) + \Gamma_0 u(kh - h) + \Gamma_1 u(kh)
\]

\[
\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} \tau (h - \frac{\tau}{2}) \\ \frac{\tau}{2} \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} (h - \tau)^2 \\ 2(h - \tau) \end{pmatrix}
\]

With \( h = 1 \) and \( \tau = 0.5 \), this gives

\[
H(z) = C(zI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 z^{-1}) = \frac{0.125(z^2 + 6z + 1)}{z(z - 2z + 1)}
\]

Order: 3
Poles: 0, 1, and 1
Zeros: \(-3 \pm \sqrt{6}\)

Examples in Matlab

\[
\begin{align*}
\text{>> Phi} & = \begin{pmatrix} 0.5 & -0.2; & 0 & 0 \end{pmatrix}; \\
\text{>> Gamma} & = \begin{pmatrix} 2; & 1 \end{pmatrix}; \\
\text{>> C} & = \begin{pmatrix} 1 & 0 \end{pmatrix}; \\
\text{>> D} & = 0; \\
\text{>> h} & = 1; \\
\text{>> H} & = \text{ss(Phi, Gamma, C, D, h);} \\
\text{>> zpk(H)} \\
\text{>> \% From cont-time transfer function to discrete-time} \\
\text{\% pulse transfer function} \\
\text{>> a = zpk('s');} \\
\text{>> G} & = 1/s^2; \\
\text{>> H} & = \text{c2d(G,h)} \\
\text{\% Another way} \\
\text{>> G} & = \text{tf([1],[1 3 2 0]);} \\
\text{>> G} & = \text{ss(G);} \\
\text{>> H} & = \text{c2d(G,h);} \\
\text{>> tf(H)}
\end{align*}
\]