Discrete time mixed $H_2/H_\infty$ control

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Introduction

Continuous time mixed $H_2/H_\infty$ control problem:


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Discrete time mixed $H_2/H_\infty$ control problem:

Some notations

- $w$ is deterministic disturbance, $w_0$ is stochastic disturbance
- $\partial D \triangleq \{ z : |z| = 1 \}$
- For stochastic processes $x$ and $y$, $x \perp y$ means $E(x - \bar{x})(y - \bar{y})^T = 0$, where $\bar{x} = E_x$, $\bar{y} = E_y$. 
Mixed $H_2/H_\infty$ control problem

Let $G(z)$ be a discrete-time system with realization

\[
\sigma x = Ax + B_0 w_0 + B_1 w + B_2 u \\
z_0 = C_0 x + D_{00} w_0 + D_{01} w + D_{02} u \\
z = C_1 x + D_{10} w_0 + D_{11} w + D_{12} u \\
y = C_2 x + D_{20} w_0 + D_{21} w
\]

which satisfies the following assumptions:

A1. $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable,

A2. $D_{02}$ has full column rank and $D_{20}$ has full row rank,

A3. \[
\begin{bmatrix}
A - zI & B_2 \\
C_0 & D_{02}
\end{bmatrix}
\]
has full column rank $\forall z \in \partial D$,

A4. \[
\begin{bmatrix}
A - zI & B_0 \\
C_2 & D_{20}
\end{bmatrix}
\]
has full row rank $\forall z \in \partial D$,

A5. $w \in P$ is a bounded power signal and $w_0 \sim GW(0, I)$ is a normalized Gaussian white noise. Moreover, $w_0(k) \perp w(j)$, $\forall k \geq j$, $w_0$ is independent of $w_0$,

A6. the initial condition $x(0) \sim G(x_0, R_0)$ is independent of $w_0$, $\forall k \geq 0$. 
Mixed $H_2/H_\infty$ control

The mixed $H_2/H_\infty$ problem: find $u^*$ and $w^*$ such that

$$J_0(w^*, u^*) \geq J_0(w, u^*), \; J_2(w^*, u^*) \leq J_2(w^*, u)$$

is solvable, if and only if, the coupled algebraic Riccati equations

$$\begin{align*}
(A + B_2 F_2)^T X_\infty \left( I - \gamma^{-2} B_1 B_1^T X_\infty \right)^{-1} (A + B_2 F_2) \\
- X_\infty + (C_1 + D_{12} F_2)^T (C_1 + D_{12} F_2) = 0
\end{align*}$$

$$\begin{align*}
(A_C + (B_1 - B_2 R_{02}^{-1} D_{02} D_{01}) H_\infty)^T X_2 (A_C \\
+ (B_1 - B_2 R_{02}^{-1} D_{02} D_{01}) H_\infty) - X_2 \\
+ (C_0 + D_{21} H_\infty)^T \tilde{R}_{02} (C_0 + D_{21} H_\infty) = 0
\end{align*}$$

$$\begin{align*}
\left(A_F + \left(B_1 - B_1 D_2^T R_{20}^{-1} D_{21}\right) H_\infty\right) Y_2 \left(I \\
+ C_2^T R_{20}^{-1} C_2 Y_2\right)^{-1} \left(A_F + \left(B_1 - B_1 D_2^T R_{20}^{-1} D_{21}\right) H_\infty\right)^T \\
- Y_2 + B_0 \tilde{R}_{20} B_0^T = 0
\end{align*}$$
Mixed $H_2/H_\infty$ control

In the three Ricaati equations,

$$F_2 = - \left( D_{02}^T D_{02} + B_2^T X_2 B_2 \right)^{-1}$$

$$\left( B_2^T X_2 (A + B_1 H_\infty) + D_{02}^T (C_0 + D_{01} H_\infty) \right)$$

$$L_2 = - \left( (A + B_1 H_\infty) Y_2 (C_2 + D_{21} H_\infty)^T + B_0 D_{20}^T \right)$$

$$\left( D_{20} D_{20}^T + (C_2 + D_{21} H_\infty) Y_2 (C_2 + D_{21} H_\infty)^T \right)^{-1}$$

$$A_{C} = A - B_2 R_{02}^{-1} D_{02}^T C_0$$

$$A_{F} = A - B_0 D_{02}^T R_{20}^{-1} C_2$$

$$R_{02} = D_{02}^T D_{02}$$

$$\tilde{R}_{02} = I - D_{02} R_{02}^{-1} D_{02}^T$$

$$R_{20} = D_{20} D_{20}^T$$

$$\tilde{R}_{20} = I - D_{20}^T R_{20}^{-1} D_{20}$$
Mixed $H_2/H_\infty$ control

The three Ricaati equations have stabilizing solutions $X_\infty \geq 0$, $X_2 \geq 0$ and $Y_2 \geq 0$.

The optimal controller $u^* = K(z)y$ has a realization given by

$$\sigma x_K = (A + B_1H_\infty + B_2F_2 + 2C_2 + L_2D_{21}H_\infty) x_K - L_2y$$

$$u^* = F_2x_K$$

Moreover the optimal value of the performance functional

$$J_2^*(w^*, u^*, w_0)$$

is

$$J_2^* = \text{tr} \left( B_0^T X_2 B_0 + D_{00}^T D_{00} \right) + \text{tr} \left( \left( B_2^T X_2 B_2 + D_{02}^T D_{02} \right) F_2 Y_2 F_2^T \right)$$
A numerical example

A discrete-time system

\[
\begin{align*}
\sigma x &= Ax + B_0 w_0 + B_1 w + B_2 u = 1.05x + w_0 + w - 0.55u \\
z_0 &= C_0 x + D_{00} w_0 + D_{01} w + D_{02} u = 0.94x + 1.36u \\
z &= C_1 x + D_{10} w_0 + D_{11} w + D_{12} u = -0.54x + 0.57u \\
y &= C_2 x + D_{20} w_0 + D_{21} w = -0.59x + 0.52w_0 + w
\end{align*}
\]

Use MATLAB to solve nonlinear equations. The optimal mixed \(H_2/H_\infty\) controller is

\[
\begin{align*}
\sigma x_K &= (A + B_1 H_\infty + B_2 F_2 + 2 C_2 + L_2 D_{21} H_\infty) x_K - L_2 y \\
&= 0.9376 x_K + 0.1406 y \\
u^* &= F_2 x_K = 0.3457 x_K
\end{align*}
\]

The optimal \(H_2\) cost is \(J_2^* = 3.42\).
Future work

- Compare the mixed $H_2/H_\infty$ performance with pure $H_2$ and $H_\infty$ controllers
- LMI, Nash game (MIMO)
- Minimize $H_\infty$ cost with constraint of $H_2$ cost.