Lecture 4: Input/output pole-placement

- Simple design procedure
- "Remove" factors in $A$ and $B$
- Shape command signal response
- Example
- "Add" factors in $R$ and $S$
- Practical aspects
- More examples
- Sensitivity

State-space design – output feedback

Problem formulation

* Process
  \[ H(z) = \frac{B(z)}{A(z)} \]
* Observer polynomial $A_o(z)$ stable
* Desired characteristic equation
  \[ A_c(z) = A_c(z)A_o(z) \quad A_c \text{ controller polynomial, stable} \]
* Controller
  \[ R(q)u(k) = T(q)u_c(k) - S(q)y(k) \]
  Causality implies
  \[ \deg R \geq \deg T \quad \deg R \geq \deg S \]

A formal solution

Closed loop system

\[ A(q)y(k) = B(q)u(k) \]
\[ R(q)u(k) = T(q)u_c(k) - S(q)y(k) \]

Desired input-output relation

\[ y = \frac{BT}{AR + BS} u_c \]
\[ = \frac{BT}{A_cA_o} = \frac{t_0B}{A_c} \]

Problem: How to determine $R$, $S$, and $T$?
Simple pole-placement design

**Data:** Model: $B(z)/A(z)$, $A(z)$ and $B(z)$ do not have any common factors. Specifications: Desired closed-loop characteristic polynomial $A_{cl}(z)$.

**Step 1.** Find $R(z)$ and $S(z)$ with $\deg S(z) \leq \deg R(z)$ such that

$$A(z)R(z) + B(z)S(z) = A_{cl}(z)$$

**Step 2.** Factor the closed-loop characteristic polynomial as $A_{cl}(z) = A_c(z)A_o(z)$, where $\deg A_o(z) \leq \deg R(z)$, and choose $T(z) = t_0A_o(z)$

where $t_0 = A_c(1)/B(1)$. The control law is

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k) \Rightarrow A_c(q)y(k) = t_0B(q)u_c(k)$$

Diophantine equation

$$A(z)X(z) + B(z)Y(z) = C(z)$$

Diophantine (Diophantus $\approx$ A.D. 300), Aryabhata, Bezout

- Two unknowns, one equation?!
- When has the Diophantine equation a (unique) solution?
- An algebraic digression

An algebraic digression

Assume $x$ and $y$ integers

$$3x + 2y = 5$$

Some solutions

$$x : \quad -5 \quad -3 \quad -1 \quad 1 \quad 3 \quad 5 \quad 7$$

$$y : \quad 10 \quad 7 \quad 4 \quad 1 \quad -2 \quad -5 \quad -8$$

General solution

$$x = x_0 + 2n \quad n \text{ integer}$$

$$y = y_0 - 3n$$

Unique solution if

$$0 \leq x < 2 \quad \text{or} \quad 0 \leq y < 3$$

No solution to

$$4x + 6y = 1$$

Integers and polynomials are rings

Main result

Diophantine equation

$$A(z)X(z) + B(z)Y(z) = C(z)$$

**Theorem**

- Solution exists if and only if greatest common factor of $A$ and $B$ also a factor in $C$
- Many solutions. If $X_0$ and $Y_0$ is a solutions then for arbitrary $Q$

$$X = X_0 + QB$$

$$Y = Y_0 - QA$$

is also a solution
- Uniqueness if

$$\deg X < \deg B \quad \text{or} \quad \deg Y < \deg A$$
### Compatibility conditions

\[ AR + BS = A_c A_o \]
\[ Ru = -S y + T u_c \]

* Causality

\[
\begin{align*}
\deg R &\geq \deg S \\
\deg R &\geq \deg T \\
\end{align*}
\]

Equality implies no delay in the controller.

* Uniqueness (Minimum degree solution)

\[
\begin{align*}
\deg S &< \deg A \\
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\deg S = \deg R = \deg T = \deg A_o = n - 1 \\
\deg A_c = n \\
\end{align*}
\]

### Solution of Diophantine equation

\[
A(z)X(z) + B(z)Y(z) = C(z)
\]

\[
(a_0 z^n + \ldots + a_n)(x_0 z^{n-1} + \ldots + x_{n-1}) + (b_0 z^n + \ldots + b_n)(y_0 z^{n-1} + \ldots + y_{n-1}) = c_0 z^{2n-1} + \ldots + c_{2n-1}
\]

Equality implies no delay in the controller.

\[
\begin{pmatrix}
    a_0 & 0 & \ldots & 0 & b_0 & 0 & \ldots & 0 \\
    a_1 & a_0 & \ldots & 0 & b_1 & b_0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{n-1} & a_{n-2} & \ldots & a_0 & b_{n-1} & b_{n-2} & \ldots & b_0 \\
    a_n & a_{n-1} & \ldots & a_1 & b_n & b_{n-1} & \ldots & b_1 \\
    0 & a_n & \ldots & a_2 & 0 & b_n & \ldots & b_2 \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & 0 & \ldots & a_n & 0 & 0 & \ldots & b_n
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{n-1}
\end{pmatrix}
\]

### Cancellation of poles and zeros

Why? Why not? Factorize

\[
A = A^+ A^- \quad B = B^+ B^-
\]

\(A^+\) and \(B^+\) are “nice” polynomials.

Cancellation of \(A^+\) and \(B^+\) implies

\[
R = B^+ \bar{R} \quad S = A^+ \bar{S} \quad T = A^+ \bar{T}
\]

\[
A_{cl} = AR + BS = A^+ B^+(A^- \bar{R} + B^- \bar{S}) = A^+ B^+ \bar{A}_{cl}
\]

Introduce (quite arbitrarily)

\[
A_c = B^+ \bar{A}_c \quad A_o = A^+ \bar{A}_o
\]

Design equation

\[
A^- \bar{R} + B^- \bar{S} = \bar{A}_{cl} = \bar{A}_c \bar{A}_o
\]

### Design procedure

Desired closed-loop characteristic equation

\[
A^- \bar{R} + B^- \bar{S} = \bar{A}_{cl} = \bar{A}_c \bar{A}_o
\]

Minimum degree solution if \(\deg \bar{S} < \deg A^- \quad \Rightarrow \quad B^+ \bar{R} u = A^+ T u_c - A^- \bar{S} y
\]

Interpretation

\[
u = \frac{A^+}{B^+} \left( \frac{T}{\bar{R}} u_c - \frac{S}{\bar{R}} y \right)
\]

Cancel some poles and zeros, and then make the design.

The simple case \(T = t_0 A_o\)

\[
\begin{align*}
\frac{B T}{A_{cl}} &= \frac{t_0 B^+ B^- A_o}{A_c A_o} = \frac{t_0 B^-}{A_c}
\end{align*}
\]
Practical limitations on $B^+$

- Don't cancel all zeros within the unit circle!
- Avoid zeros on the negative real axis
- Avoid poorly damped zeros
- Cancel only in shaded area D

Separation of disturbance and command resp.

Compare state-feedback design and feedforward from reference signal. Desired command signal response

$$y_m = H_m u_c = \frac{B_m}{A_m} u_c$$

Limitation: $B_m = \bar{B}_m B^-$ Try the controller

$$R = A_m B^+ \bar{R} \quad S = A_m A^+ \bar{S} \quad T = \bar{B}_m \bar{A}_o \bar{A}_c A^+$$

If common factors between $A_m$ and $\bar{A}_c$, cancel before implementation of the controller

$$\frac{B T}{A R + B S} = \frac{B_+ B^- B_m \bar{A}_o \bar{A}_c A^+}{A_+ A_- A_m B^+ \bar{R} + B_+ B^- A_m A^+ \bar{S}} = \frac{B \bar{B}_m \bar{A}_o \bar{A}_c}{A_m (A^- \bar{R} + B^- \bar{S})}$$

Alternativ formulation

Feedforward and feedback gives the closed loop system

$$y = \frac{B_m}{A_m} u_c$$

The combined generation of $y_m$ and $u_{ff}$ requires great care.

Example – Motor

$$H(z) = \frac{K(z - b)}{(z - 1)(z - a)} \quad b < 0! \quad H_m(z) = \frac{z(1 + p_1 + p_2)}{z^2 + p_1 z + p_2}$$

Cancels the zero $B^+ = z - b$, $B^- = K$, $\bar{B}_m = B_m / K$, $A^+ = 1$, $A_o = 1$, and $\bar{A}_c = A_m$.

Control law using $A \bar{R} + B^- \bar{S} = A_m$, $\deg S = 1$, $\deg \bar{R} = 0$

a) $h = 0.25$, b) $h = 1$
Motor example – No cancellation

\[ H_m(z) = \frac{1 + p_1 + p_2}{1 - b} \frac{z - b}{z^2 + p_1 z + p_2} \]

\[ B^+ = 1, B^- = K(z - b), A^+ = 1, A_c = A_m, A_o = z, \text{ and} \]

\[ \tilde{B}_m = \frac{1 + p_1 + p_2}{K(1 - b)} \]

Control law using \( AR + B^- S = A_m A_o, \deg S = 1, \deg R = 1 \)

\[ u(k) = t_0 u_c(k) - s_0 y(k) - s_1 y(k - 1) - r_1 u(k - 1) \]

Note: The same structure as for cancellation

a) \( h = 0.25 \), b) \( h = 1 \)

"Add" factors in \( R \) ans \( S \)

\[ Ru = T u_c - S y \]

Input \[ u_c \]

Output \[ v \]

\[ \Sigma \]

\[ \frac{B}{A} \]

Output \[ e \]

Input \[ y \]

\[ \Sigma \]

\[ x = \frac{BT}{AR + BS} u_c + \frac{BR}{AR + BS} v - \frac{BS}{AR + BS} e \]

\[ u = \frac{AT}{AR + BS} u_c - \frac{AS}{AR + BS} v - \frac{AS}{AR + BS} e \]

- Integrators \( R = B^+ R_d \tilde{R}, R_d = (z - 1)^l \)
- Notch filter \( S = A^+ S_d \tilde{S} \) where e.g.

\[ S_d(z) = z^2 - 2z \cos \omega h + 1 \]

Full design algorithm

Data: \( B/A, B_m/A_m, R_d, S_d, \text{ and } A_{cl} \)

Step 1: Factor \( A = A^- A^+, B = B^- B^+, B_m = B^- \tilde{B}_m \), \( A_{cl} = A^+ B^+ A_m \tilde{A}_{cl} \)

Step 2: Solve

\[ A^- R_d \tilde{R} + B^- S_d \tilde{S} = \tilde{A}_{cl} \]

\( \deg \tilde{S} = \deg A^- R_d - 1 \)

Step 3: Control law

\[ Ru = T u_c - S y R = A_m B^+ R_d \tilde{R}, \quad S = A_m A^+ S_d \tilde{S} \quad T = \tilde{B}_m A^+ \tilde{A}_{cl} \]

Degree conditions:

\[ \deg A_m - \deg B_m \geq \deg A - \deg B = d \]

\[ \deg A_{cl} = (\deg AR = \deg A + \deg S) \]

\[ = 2 \deg A + \deg A_m + \deg R_d + \deg S_d - 1 \]

Practical aspects

* Solution of the Diophantine equation
* Zero cancellations
* How to choose \( A_c \) and \( A_o \)?
* Magnitude of \( u \)

\[ y = H u \]

\[ y = H_m u_c \Rightarrow u = H_m H u_c \]

* Selection of \( h \) \( (A_o A_c B^+) \)

\[ \omega h = 0.2 - 0.6 \]

* Response to load and noise
* Influence of observer polynomial
Observer polynomial

Consider

\[ H(z) = \frac{0.1}{z - 1} \quad H_m(z) = \frac{0.2}{z - 0.8} \]

\[ A_o(z) = z - \alpha \quad \text{Transmission to } x \text{ from a) Load } v \text{ and b) Measurement error } e \]

\( \alpha = 1 \) (full), \( \alpha = 0.9 \) (dashed), \( \alpha = 0.5 \) (dash-dotted), \( \alpha = 0 \) (dotted)

Harmonic oscillator

Process model

\[ G(s) = \frac{\omega_0^2}{s^2 + \omega_0^2} \quad \omega_0 = 1 \]

Sampled pulse-transfer operator

\[ H(q) = \frac{(1 - \beta)(q + 1)}{q^2 - 2\beta q + 1} = \frac{B(q)}{A(q)} \quad \beta = \cos(\omega_0 h) \]

Specifications (nominal design)

- No zero cancellation
- \( A_e : \quad s^2 + 2\zeta \omega s + \omega^2 = 0 \quad \omega = 1.5 \quad \zeta = 0.7 \)
- \( A_o : \quad s^2 + 2\zeta_{obs} \omega_{obs} s + \omega_{obs}^2 = 0 \quad \omega_{obs} = 3 \quad \zeta_{obs} = 0.7 \)
- Sampling interval \( h = 0.2 \)

Harmonic oscillator cont’d

Nominal design a) Without, b) With integrator

Changing observer dynamics

a) \( \omega_{obs} = 4 \) b) \( \omega_{obs} = 8 \)
Harmonic oscillator cont’d

Changing sampling interval
Nominal $\omega_{obs} h = 0.6$ a) $h = 0.1$ b) $h = 1$

(a) $h = 0.6$

(b) $h = 0.1$

(b) $h = 1$

Harmonic oscillator cont’d

Antialiasing filter 6th order Bessel
Nominal design gives unstable system
Approximate the filter with a delay and redo the design

Robot mechanism example

Antialiasing filter, second order with $\omega_f = 2$

$$A(q) = \left( q^2 - 0.7505q + 0.2466 \right) \left( q^2 - 1.7124q + 0.9512 \right) (q - 1)$$

$$B(q) = 0.1420 \cdot 10^{-3} (q + 12.1314)(q + 1.3422)(q + 0.2234)(q - 0.0023)$$

Robot mechanism example

Poles and zeros
**Notch filter design**

Sample, \( h = 0.5, A_c(s) = (s^2 + 2\zeta_m \omega_m s + \omega_m^2)(s + \alpha_1 \omega_m) \) and keep the antialiasing dynamics

- \( \text{deg} A_o = 2 \) Same poles as \( A_f \)
- Include the oscillatory part

\[
A^+(z) = z^2 - 1.712z + 0.9512
\]

\( AR + BS = A^+ A_c A_o \)

4th order controller \( \Rightarrow \) 9th order closed loop system

\( A_n \) factor in \( A \) but not in \( B \) \( \Rightarrow \) factor in \( S \)

**Active damping**

Damp the oscillatory modes, \( \zeta_p = 0.05 \rightarrow 0.7 \)

**Comparison**

State feedback design and full observer (No antialiasing filter)

**Sensitivity**

The design is done for \( H = B/A \) but the true system is \( H^0 = B^0/A^0 \)

Problem: How sensitive is the closed loop system?

Theorem

The closed loop system is stable if

\[
\frac{|H(z) - H^0(z)|}{|H(z)|} \leq \frac{1}{|H_m(z)|} \frac{|H_{ff}(z)|}{|H_{fb}(z)|} = \frac{1}{|H_m(z)|} \left| \frac{T}{S} \right|
\]

for \( |z| = 1 \)

Right hand side depends on known quantities!
Robot mechanism
Sensitivity function for notch design and active damping design
\[ S = \frac{AR}{A_{cl}} \]

Notch (full), active (dashed)

Smith-predictor
One way of reducing the effect of delays
- Design a regulator \( G_r \) as if there is no delay.
- Find \( G'_r \) such that
  \[ \frac{G_r G_p}{1 + G_r G_p} = \frac{e^{sT} G'_r G_p}{1 + G'_r G_p} \]
- Good for discrete-time systems

Smith-predictor – Example
First order system with time-delay
\[ y(k + 1) = 0.37y(k) + 0.63u(k - 2) \]
No delay: PI with \( K = 0.4 \) and \( T_i = 0.4 \)
Smith pred. (full) and PI-contr, \( K = 0.1 \), \( T_i = 0.5 \) (dashed)

Summary
- Convenient method for design
- Design parameters \( A_c, A_o, \) and \( h \)
- Relate \( A_c \) and \( A_o \) to physical process
- Be careful with zero cancellations
- Many other methods can be interpreted as pole-placement