### Lecture 2: $z$-transform and I/O models

- Shift operator
- I/O models
- Direct sampling
- $z$-transform
- Poles and zeros
- Selection of sampling interval
- Frequency response of sampled-data systems
- Lyapunov theory for discrete-time systems

### Shift-operator

**Forward shift operator**

\[ qf(k) = f(k + 1) \]

**Backward shift (delay) operator**

\[ q^{-1}f(k) = f(k - 1) \]

The range of the shift operator is double infinite sequences

Compare with the differential operator

\[ p = \frac{d}{dt} \]

### Shift-operator calculus

\[
y(k + na) + a_1y(k + na - 1) + \cdots + a_{na}y(k) = b_0u(k + nb) + \cdots + b_{nb}u(k)
\]

where $na \geq nb$. Using the shift operator gives

\[
(q^{na} + a_1q^{na-1} + \cdots + a_{na})y(k) = (b_0q^{nb} + \cdots + b_{nb})u(k)
\]

Introduce the polynomials

\[
A(z) = z^{na} + a_1z^{na-1} + \cdots + a_{na}
\]

\[
B(z) = b_0z^{nb} + b_1z^{nb-1} + \cdots + b_{nb}
\]

the difference equation can be written as

\[
A(q)y(k) = B(q)u(k)
\]

\[
y(k) = \frac{B(q)}{A(q)}u(k)
\]

### Reciprocal polynomials

\[
y(k + na) + a_1y(k + na - 1) + \cdots + a_{na}y(k) = b_0u(k + nb) + \cdots + b_{nb}u(k)
\]

can be written as

\[
y(k) + a_1y(k - 1) + \cdots + a_{na}y(k - na) = b_0u(k - d) + \cdots + b_{nb}u(k - d - nb)
\]

Pole excess $d = na - nb$

Reciprocal polynomial

\[
A^*(z) = 1 + a_1z + \cdots + a_{na}z^{na} = z^{na}A(z^{-1})
\]

The system description in the backward shift operator

\[
A^*(q^{-1})y(k) = B^*(q^{-1})u(k - d)
\]

\[
y(k) = \frac{B^*(q^{-1})}{A^*(q^{-1})}u(k - d)
\]
**Pulse-transfer function operator**

State-space system

\[ x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k) \]

Use the shift operator

\[ (qI - \Phi)x(k) = \Gamma u(k) \]

Eliminate \( x(k) \)

\[ y(k) = Cx(k) + Du(k) = (C(qI - \Phi)^{-1}\Gamma + D)u(k) \]

Pulse-transfer operator

\[ H(q) = C(qI - \Phi)^{-1}\Gamma + D \]

In the backward-shift operator

\[ H^*(q^{-1}) = C(I - q^{-1}\Phi)^{-1}q^{-1}\Gamma + D = H(q) \]

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**Poles, zeros, and system order**

\[ H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)} \]

Poles: \( A(q) = 0 \)
Zeros: \( B(q) = 0 \)
System order: \( \deg A(q) \)

Important to use the forward shift operator for poles/zeros, system order, and stability.

The backward shift operator is suited for causality considerations.

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**SISO systems**

\[ H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)} \]

If no common factors

\[ \deg A = n \]
\[ A(q) = \det[qI - \Phi] \]

and

\[ y(k) + a_1y(k-1) + \cdots + a_ny(k-n) = b_0u(k) + \cdots + b_nu(k-n) \]

where \( a_i \) are the coefficients of the characteristic polynomial of \( \Phi \).

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**Example – Double integrator with delay**

\( h = 1 \) and \( \tau = 0.5 \) gives

\[ \Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} 0.375 \\ 0.5 \end{pmatrix} \quad \Gamma_0 = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix} \]

Then

\[ H(q) = C(qI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 q^{-1}) \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{q - 1}{(q - 1)^2} \begin{pmatrix} 0.125 + 0.375q^{-1} \\ 0.5 + 0.5q^{-1} \end{pmatrix} \]

\[ = \frac{0.125(q^2 + 6q + 1)}{q(q^2 - 2q + 1)} = \frac{0.125(q^{-1} + 6q^{-2} + q^{-3})}{1 - 2q^{-1} + q^{-2}} \]

Order: 3

Poles: 0, 1, and 1

Zeros: \( -3 \pm \sqrt{8} \)
How to get $H(q)$ from $G(s)$?

Use Table 2.1
Zero-order hold sampling of a continuous-time system, $G(s)$.

$$H(q) = \frac{b_1q^{n-1} + b_2q^{n-2} + \cdots + b_n}{q^n + a_1q^{n-1} + \cdots + a_n}$$

<table>
<thead>
<tr>
<th>$G(s)$</th>
<th>$H(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{s}$</td>
<td>$\frac{h}{q-1}$</td>
</tr>
<tr>
<td>$\frac{1}{s^2}$</td>
<td>$\frac{h^2(q+1)}{2(q-1)^2}$</td>
</tr>
<tr>
<td>$\frac{a}{s+a}$</td>
<td>$\frac{1-\exp(-ah)}{q-\exp(-ah)}$</td>
</tr>
</tbody>
</table>

**z-transform**

Definition of z-transform
Consider the discrete-time signal \( \{ f(kh) : k = 0, 1, \ldots \} \).

$$Z(f(kh)) = F(z) = \sum_{k=0}^{\infty} f(kh)z^{-k}$$

The inverse transform is given by

$$f(kh) = \frac{1}{2\pi i} \oint F(z)z^{k-1} dz$$

where the contour of integration encloses all singularities of \( F(z) \). Maps a semi-infinite time sequence into a function of a complex variable

**Example**

Let \( y(kh) = kh \) for \( k \geq 0 \). Then

\[
Y(z) = 0 + hz^{-1} + 2hz^{-2} + \cdots \\
= h(z^{-1} + 2z^{-2} + \cdots) \\
= \frac{hz}{(z-1)^2}
\]

- Similarities with Laplace transform
- Common in applied mathematics
- How the theory of sampled-data systems started

**Properties of z-transform**

1. Definition.

\[ F(z) = \sum_{k=0}^{\infty} f(kh)z^{-k} \]

2. Time shift.

\[
Zq^{-n}f = z^{-n}F \\
Z\{q^n f\} = z^n(F - F_1) \\
\text{where } F_1(z) = \sum_{j=0}^{n-1} f(jh)z^{-j}
\]

3. Initial value theorem.

4. Final-value theorem.

5. Convolution.

\[
Z(f * g) = \sum_{n=0}^{k} f(n)g(k - n) = (Zf)(Zg)
\]
### Pulse-transfer function

\[ x(k + 1) = \Phi x(k) + \Gamma u(k) \]
\[ y(k) = C x(k) + D u(k) \]

Take the z-transform of both sides

\[ z \left( \sum_{k=0}^{\infty} z^{-k} x(k) - x(0) \right) = \sum_{k=0}^{\infty} \Phi z^{-k} x(k) + \sum_{k=0}^{\infty} \Gamma z^{-k} u(k) \]

Hence

\[ z(X(z) - x(0)) = \Phi X(z) + \Gamma U(z) \]
\[ X(z) = (zI - \Phi)^{-1}(zx(0) + \Gamma U(z)) \]
\[ Y(z) = C(zI - \Phi)^{-1}zx(0) + (C(zI - \Phi)^{-1}\Gamma + D)U(z) \]

**Pulse-transfer function**

\[ H(z) = C(zI - \Phi)^{-1}\Gamma + D \]

### Why both \( q \) and \( z \)?

- Could be sufficient with only the shift operator \( q \)
- Many books contain the \( z \)-transform
- Must be aware of the difficulties with \( z \)-transform
- Remember \( q \) operator and \( z \) complex variable

### Calculation of \( H(z) \) given \( G(s) \) using \( z \)-transform tables

1. Determine the step response of the system with the transfer function \( G(s) \).
2. Determine the corresponding \( z \)-transform of the step response using the table.
3. Divide by the \( z \)-transform of the step function.

\[ Y(s) = \frac{G(s)}{s} \rightarrow \tilde{Y} = \mathcal{Z}(\mathcal{L}^{-1}Y) \]
\[ \rightarrow H(z) = (1 - z^{-1})\tilde{Y}(z) \]

### A warning

!!!Use the \( z \)-transform tables correctly!!!!

<table>
<thead>
<tr>
<th>( f(kh) )</th>
<th>( \mathcal{L}f(t) )</th>
<th>( \mathcal{Z}f(kh) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(k) ) (pulse)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{s} )</td>
<td>( \frac{z}{z - \frac{1}{h}} )</td>
</tr>
<tr>
<td>( kh )</td>
<td>( \frac{1}{s^2} )</td>
<td>( \frac{(z - 1)^2}{h^2z(z + 1)} )</td>
</tr>
<tr>
<td>( \frac{1}{2}(kh)^2 )</td>
<td>( \frac{1}{s^3} )</td>
<td>( \frac{h^2z(z + 1)}{2(z - 1)^3} )</td>
</tr>
<tr>
<td>( e^{-kh/T} )</td>
<td>( \frac{1}{1 + st} )</td>
<td>( \frac{z - e^{-h/T}}{z(1 - e^{-h/T})} )</td>
</tr>
<tr>
<td>( 1 - e^{-kh/T} )</td>
<td>( \frac{1}{s(1 + st)} )</td>
<td>( \frac{z(1 - e^{-h/T})}{(z - 1)(z - e^{-h/T})} )</td>
</tr>
</tbody>
</table>

**Warning.** Notice that \( \mathcal{Z}f \) in the table does not give the zero-order-hold sampling of a system with the transfer function \( \mathcal{L}f \).
Double integrator – Sampling using table

Transfer function \( G(s) = 1/s^2 \)
Introduce the step
\[
Y(s) = \frac{1}{s^3}
\]
Use the table
\[
\tilde{Y} = Z(L^{-1}Y) = \frac{h^2(z+1)}{2(z-1)^3}
\]
Get the pulse transfer function
\[
H(z) = (1-z^{-1})\tilde{Y}(z) = \frac{h^2(z+1)}{2(z-1)^2}
\]

Formula for \( H(z) \)

The following formula can be derived:
\[
H(z) = \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sh}}{z-e^{sh}} \frac{G(s)}{s} \, ds
\]

If \( G(s) \) goes to zero at least as fast as \(|s|^{-1}\) for a large \( s \) and has distinct poles (none at the origin)
\[
H(z) = \sum_{s=s_i} \frac{1}{z-e^{sh}} \text{Res} \left\{ \frac{e^{sh} - 1}{s} \right\} G(s)
\]
where \( s_i \) are the poles of \( G(s) \)

Multiple poles influence the calculations of the residues.

Modified \( z \)-transform

Can be used to determine intersample behavior

Definition: Modified \( z \)-transform
\[
\tilde{F}(z,m) = \sum_{k=0}^{\infty} z^{-k} f(kh - h + mh), \quad 0 \leq m \leq 1
\]
The inverse transform is given by
\[
f(nh - h + mh) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{F}(z,m) z^{n-1} \, dz
\]
\( \Gamma \) encloses all singularities of the integrand

Interpretation of poles and zeros

Poles:
- A pole \( z = a \) is associated with the time function \( z(k) = a^k \)
- A pole \( z = a \) is an eigenvalue of \( \Phi \)

Zeros:
- A zero \( z = a \) implies that the transmission of the input \( u(k) = a^k \) is blocked by the system
- A zero is related to how inputs and outputs are coupled to the states
Transformation of poles $\lambda_i(\Phi) = e^{\lambda(A)h}$

New evidence of alias problem

$z = e^{sh}$

Several points in the $s$-plane is mapped into the same point in the $z$-plane.
The map is not bijective

Sampling of a second order system

More difficult than poles
In general, more sampled zeros than continuous
For short sampling periods $z_i \approx e^{s_i h}$
For large $s$ then $G(s) \approx s^{-d}$
where $d = \deg A(s) - \deg B(s)$
The $r = d - 1$ sampling zeros go to the zeros of the polynomials $Z_d$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$Z_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>$z + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$z^2 + 4z + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$z^3 + 11z^2 + 11z + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$z^4 + 26z^3 + 66z^2 + 26z + 1$</td>
</tr>
</tbody>
</table>
**Systems with unstable inverse**

Continuous-time system is nonminimum phase if it has right half-plane zeros or time delays.

A discrete-time system is in many books defined to be nonminimum phase if it has zeros outside the unit disc.

We will use the following notation:

**Definition – Unstable inverse**

A discrete-time system has an unstable inverse if it has zeros outside the unit disc.

Nonminimum phase \iff Unstable inverse

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**Selection of sampling period**

Number of samples per rise time

\[ N_r = \frac{T_r}{h} \approx 4 - 10 \]

The rise times of the signals are \( T_r = 1 \).

a) \( N_r = 1 \), b) \( N_r = 2 \),

c) \( N_r = 4 \), d) \( N_r = 8 \)

---

**Second order system**

\[ N_r = \frac{T_r}{h} \approx 4 - 10 \]

Corresponds to (for dominating modes)

\[ \omega_0 h \approx 0.2 - 0.6 \]

\( \zeta = 0.5, \omega_0 = 1.83 \) gives \( T_r = 1 \);

a) \( h = 0.125 (\omega_0 h = 0.23) \)

b) \( h = 0.25 (\omega_0 h = 0.46) \)

c) \( h = 0.5 (\omega_0 h = 0.92) \)

d) \( h = 1.0 (\omega_0 h = 1.83) \)

---

**Pole-zero variation with \( h \)**

\[ G(s) = \frac{1}{(s + 1)(s^2 + s + 1)} \]

\( h = 0.0001, 0.2, 0.5, 1, 2, \) and 3
Nyquist and Bode diagrams

Nyquist curve: $H(e^{j\omega h})$ for $\omega h \in [0, \pi]$, i.e. up to $\omega_N$

- Periodic
- Interpretation
- Higher order harmonics
- Discuss more in connection with Chapter 7

Example

$$G(s) = \frac{1}{s^2 + 1.4s + 1}$$

Zero-order hold sampling
$h = 0.4$

$$H(z) = \frac{0.066z + 0.055}{z^2 - 1.450z + 0.571}$$

Continuous-time (dashed), discrete-time (full)

Lyapunov theory

Consider the system

$$x(k + 1) = f(x(k)), \quad f(0) = 0$$

Monotonic convergence $\|x(k + 1)\| < \|x(k)\|$ a too strong condition for stability

Find other "norm", a Lyapunov function $V(x)$

- $V(x)$ is continuous in $x$ and $V(0) = 0$
- $V(x)$ is positive definite
- $\Delta V(x) = V(f(x)) - V(x)$ is negative definite
- $V(x) \to \infty$, $|x| \to \infty$

Existence of Lyapunov function implies asymptotic stability for the solution $x = 0$
**Geometric interpretation**

\[ x(k + 1) = f(x(k)), \quad f(0) = 0 \]

**Linear system**

\[ x(k + 1) = \Phi x(k) \]

\[ V(x) = x^T P x \quad P > 0 \]

\[ \Delta V(x) = V(\Phi x) - V(x) = x^T \Phi^T P \Phi x - x^T P x \]

\[ = x^T (\Phi^T P \Phi - P) x = -x^T Q x \]

\( V \) is a Lyapunov function iff there exists a \( P > 0 \) that satisfies the Lyapunov equation

\[ \Phi^T P \Phi - P = -Q \quad Q > 0 \]

**Summary**

- Piecewise constant input and periodic sampling gives time-invariant discrete-time system
- Solution of the system equation, \( \lambda(\Phi) \)
- Shift operator \( q \) and pulse transfer operator
- \( z \)-transform and pulse transfer function
- Be careful with \( z \)-transform tables
- Poles, zeros, and system order
- Selection of sampling period

\[ N_r = \frac{T_r}{h} \approx 4 - 10 \]

\[ \omega_0 h \approx 0.2 - 0.6 \]

- Frequency function

**Example**

\[ \Phi = \begin{pmatrix} 0.4 & 0 \\ -0.4 & 0.6 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]